Linear Maps

Objectives:

1. Introducing linear maps represented as matrices.
2. Determining matrix representations of a given linear map.
3. Introducing common linear maps, such as rotation and scaling.

Recall: Change-of-Basis

Let $B = \{\vec{\beta}_1, \vec{\beta}_2\}$ be a set of basis vectors in $\mathbb{R}^2$.

$$\vec{\beta}_1 = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} \text{ and } \vec{\beta}_2 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \xrightarrow{\text{matrix}} A = \begin{array}{cc} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \\ \;\;\vec{\beta}_1 \quad\quad \vec{\beta}_2 \end{array}$$

these are basis vectors because its linearly independent and span the vector space.
Suppose we have a vector using the standard basis $\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

We want to find a vector that changes the coordinates of $\vec{v}$ onto the basis B.

$$\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 \longleftarrow \text{Rep}_B(\vec{v}) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B$$

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \longleftarrow \text{written as a linear combination}$$

or

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \longleftarrow \text{written as a matrix-vector equation}$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} \sqrt{2}/2 & \sqrt{2}/2 & -3 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 4 \end{array}\right] \longleftarrow \text{augmented matrix}$$

$$\downarrow \text{RREF}$$

$$\begin{array}{c} \;c_1 \quad c_2 \\ \left[\begin{array}{cc|c} 1 & 0 & -7/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} \end{array}\right] \end{array} \longrightarrow \text{Rep}_B(\vec{v}) = \begin{bmatrix} -7/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}_B$$

$c_1 = -7/\sqrt{2}$
$c_2 = 1/\sqrt{2}$

Linear Maps

* Let $f: V \to W$ be a linear map (homomorphism).
Suppose that the set of vectors $B = \{\vec{\beta}_1, \vec{\beta}_2, \dots, \vec{\beta}_n\}$ is the bases for vector space V of dimension n.
Then, for any vector

$$\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \dots + \vec{\beta}_n,$$

the transformation is given by

$$f(\vec{v}) = c_1 f(\vec{\beta}_1) + c_2 f(\vec{\beta}_2) + \dots + c_n f(\vec{\beta}_n),$$

where the set of vectors $D = \{f(\vec{\beta}_1), f(\vec{\beta}_2), \dots, f(\vec{\beta}_n)\}$ is the bases for the vector space W of dimension m.

* Example 1: Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with bases

$$B = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\} \text{ and } D = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The transformation f is determined by this set of action;

The transformation f is determined by this set of action;

$$\begin{bmatrix}2\\0\end{bmatrix} \longrightarrow \begin{bmatrix}1\\1\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1\\4\end{bmatrix} \longrightarrow \begin{bmatrix}1\\2\\0\end{bmatrix}.$$

The goal is to find the matrix representation of f.

- Here, we notice that the domain vectors $\begin{bmatrix}2\\0\end{bmatrix}$ and $\begin{bmatrix}1\\4\end{bmatrix}$ are the bases.

  So, let $\vec{\beta}_1 = \begin{bmatrix}2\\0\end{bmatrix}$ and $\vec{\beta}_2 = \begin{bmatrix}1\\4\end{bmatrix}$.

- Next, we represent the output vectors with respect to the D basis vectors.

  - $$\begin{bmatrix}1\\1\\1\end{bmatrix} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\-2\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\1\end{bmatrix} \longleftarrow \operatorname{Rep}_D(f(\vec{\beta}_1)) = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}_D$$

    $$\left[\begin{array}{ccc|c}1&0&1&1\\0&-2&0&1\\0&0&1&1\end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c}\begin{array}{ccc}c_1&c_2&c_3\end{array}\\ \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&-1/2\\0&0&1&1\end{array}\right]\end{array} \longrightarrow \operatorname{Rep}_D(f(\vec{\beta}_1)) = \begin{bmatrix}0\\-1/2\\1\end{bmatrix}_D$$

    $c_1 = 0,\ c_2 = -1/2,\ c_3 = 1$

  - $$\begin{bmatrix}1\\2\\0\end{bmatrix} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\-2\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\1\end{bmatrix} \longleftarrow \operatorname{Rep}_D(f(\vec{\beta}_2)) = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}_D$$

    $$\left[\begin{array}{ccc|c}1&0&1&1\\0&-2&0&2\\0&0&1&0\end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&0\end{array}\right] \longleftarrow \operatorname{Rep}_D(f(\vec{\beta}_2)) = \begin{bmatrix}1\\-1\\0\end{bmatrix}_D$$

- We know that $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2$ for any $\vec{v}$.

  So, $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2$ where $\operatorname{Rep}_B(\vec{v}) = \begin{bmatrix}c_1\\c_2\end{bmatrix}_B$,

$$\begin{aligned}
f(\vec{v}) &= f(c_1\vec{\beta}_1 + c_2\vec{\beta}_2)\\
&= c_1 f(\vec{\beta}_1) + c_2 f(\vec{\beta}_2)\\
&= c_1 f\left(\begin{bmatrix}2\\0\end{bmatrix}\right) + c_2 f\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\\
&= c_1 \underbrace{\left(0\begin{bmatrix}1\\0\\0\end{bmatrix} + (-1/2)\begin{bmatrix}0\\-2\\0\end{bmatrix} + 1\begin{bmatrix}1\\0\\1\end{bmatrix}\right)}_{\operatorname{Rep}_D\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1/2\\1\end{bmatrix}_D} + c_2 \underbrace{\left(1\begin{bmatrix}1\\0\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\-2\\0\end{bmatrix} + 0\begin{bmatrix}1\\0\\1\end{bmatrix}\right)}_{\operatorname{Rep}_D\left(\begin{bmatrix}1\\2\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\0\end{bmatrix}_D}
\end{aligned}$$

$$f(\vec{v}) = \underbrace{(0c_1 + 1c_2)\begin{bmatrix}1\\0\\0\end{bmatrix} + (-(1/2)c_1 - 1c_2)\begin{bmatrix}0\\-2\\0\end{bmatrix} + (1c_1 + 0c_2)\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\operatorname{Rep}_D(f(\vec{v})) = \begin{bmatrix}0c_1 + 1c_2\\(-1/2)c_1 - 1c_2\\1c_1 + 0c_2\end{bmatrix}_D}$$

— Thus, if $Rep_B(\vec{v}) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B$, then $Rep_D(f(\vec{v})) = \begin{bmatrix} c_2 \\ (-1/2)c_1 - c_2 \\ c_1 \end{bmatrix}_D$.

check: • $\vec{v} = \vec{\beta}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 2 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \longleftarrow Rep_B\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 2 & 1 & 2 \\ 0 & 4 & 0 \end{array}\right] \xrightarrow{RREF} \begin{array}{c} \begin{matrix} c_1 & c_2 \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \end{array} \longrightarrow Rep_B\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_B \longleftarrow \text{this is } \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$c_1 = 1$
$c_2 = 0$

$$Rep_D(f(\vec{\beta}_1)) = \begin{bmatrix} 0 \\ (-1/2)(1) - 0 \\ 1 \end{bmatrix}_D = \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}_D$$

$$\downarrow$$

check:
$$0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1/2) \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

• $\vec{v} = \vec{\beta}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \longleftarrow Rep_B\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 4 & 4 \end{array}\right] \xrightarrow{RREF} \begin{array}{c} \begin{matrix} c_1 & c_2 \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \end{array} \longrightarrow Rep_B\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_B \longleftarrow \text{this is } \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$c_1 = 0$
$c_2 = 1$

$$Rep_D(f(\vec{\beta}_2)) = \begin{bmatrix} 1 \\ (-1/2)0 - 1 \\ 0 \end{bmatrix}_D = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}_D$$

$$\downarrow$$

check:
$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \checkmark$$

• for any $\vec{v}$; say $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \longleftarrow Rep_B\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 4 & 1 \end{array}\right] \xrightarrow{RREF} \begin{array}{c} \begin{matrix} c_1 & c_2 \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 0 & 3/8 \\ 0 & 1 & 1/4 \end{array}\right] \end{array} \longrightarrow Rep_B\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 3/8 \\ 1/4 \end{bmatrix}_B$$

$c_1 = 3/8$
$c_2 = 1/4$

$$Rep_D\left(f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right) = \begin{bmatrix} 1/4 \\ (-1/2)(3/8) - 1/4 \\ 3/8 \end{bmatrix}_D = \begin{bmatrix} 1/4 \\ -7/16 \\ 3/8 \end{bmatrix}_D$$

$$\downarrow$$

check:
$$(1/4) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-7/16) \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + (3/8) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (1/4)(1) + (7/16)(0) + (3/8)(1) \\ (1/4)(0) + (-7/16)(-2) + (3/8)(0) \\ (1/4)(0) + (-7/16)(0) + (3/8)(1) \end{bmatrix} = \begin{bmatrix} 5/8 \\ 7/8 \\ 3/8 \end{bmatrix} \checkmark$$

— We can represent f as a matrix,

$$\begin{bmatrix} | & | \\ Rep_D(f(\vec{\beta}_1)) & Rep_D(f(\vec{\beta}_2)) \\ | & | \end{bmatrix}_{B,D} = Rep_{B,D}(f)$$

$$\begin{bmatrix} 0 & 1 \\ -1/2 & -1 \\ 1 & 0 \end{bmatrix}_{B,D} = Rep_{B,D}(f)$$

So that

$$Rep_{B,D}(f)\, Rep_B(\vec{v}) = Rep_D(f(\vec{v}))$$

or

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -1/2 & -1 \\ 1 & 0 \end{bmatrix}_{B,D}}_{2\times 3} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B}_{2\times 1} = \underbrace{\begin{bmatrix} c_2 \\ (-1/2)c_1 - c_2 \\ c_1 \end{bmatrix}_D}_{3\times 1} .$$

$$A\vec{x} = \vec{b}$$

$f: \mathbb{R}^2 \to \mathbb{R}^3$ where f is $A = \begin{bmatrix} 0 & 1 \\ -1/2 & -1 \\ 1 & 0 \end{bmatrix}_{B,D}$.

* Example 2: Consider the same transformation in Example 1.

The transformation $f: \mathbb{R}^2 \to \mathbb{R}^3$ where f is $A = \begin{bmatrix} 0 & 1 \\ -1/2 & -1 \\ 1 & 0 \end{bmatrix}_{B,D} = Rep_{B,D}(f)$ requires us to input a vector in standard basis vectors and represent it with respect to B, $Rep_B(\vec{v})$. Then, when A is applied, the output vector is $Rep_D(f(\vec{v}))$, where you need to recover $f(\vec{v})$ in standard basis vectors by a change of basis.

We can create a matrix represtation of f using the standard basis vectors.

— Represent $\vec{e_1}$ & $\vec{e_2}$ with respect to B.

- $\vec{e_1} = c_1\vec{\beta_1} + c_2\vec{\beta_2}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 4 \end{bmatrix} \longleftarrow Rep_B(\vec{e_1}) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 4 & 0 \end{array}\right] \xrightarrow{RREF} \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{array}\right] \end{array} \longrightarrow Rep_B(\vec{e_1}) = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}_B$$

$c_1 = 1/2$

$c_2 = 0$

- $\vec{e_2} = c_1\vec{\beta_1} + c_2\vec{\beta_2}$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 4 \end{bmatrix} \longleftarrow Rep_B(\vec{e_2}) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 4 & 1 \end{array}\right] \xrightarrow{RREF} \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & -1/8 \\ 0 & 1 & 1/4 \end{array}\right] \end{array} \longrightarrow Rep_B(\vec{e_2}) = \begin{bmatrix} -1/8 \\ 1/4 \end{bmatrix}_B$$

$c_1 = -1/8$

$c_2 = 1/4$

— Next, we apply the transformation using the action.

- $\vec{e_1} = (1/2)\vec{\beta_1} + 0\,\vec{\beta_2} \longleftarrow Rep_B(\vec{e_1}) = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}_B$

$f(\vec{e_1}) = (1/2)f(\vec{\beta_1}) + 0\,f(\vec{\beta_2})$

$$\downarrow$$

$$= (1/2)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$f(\vec{e}_1)=\begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix}\longrightarrow\begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix}=c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\-2\\0\end{bmatrix}+c_3\begin{bmatrix}1\\0\\1\end{bmatrix}\longleftarrow \mathrm{Rep}_D(f(\vec{e}_1))=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}_D$$

$$\mathrm{Rep}_e(f(\vec{e}_1))=\begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix}_e$$

$$\left[\begin{array}{ccc|c}1&0&1&1/2\\0&-2&0&1/2\\0&0&1&1/2\end{array}\right]\xrightarrow{RREF}\begin{array}{cccc}c_1&c_2&c_3&\\ \end{array}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&-1/4\\0&0&1&1/2\end{array}\right]\longrightarrow \mathrm{Rep}_D(f(\vec{e}_1))=\begin{bmatrix}0\\-1/4\\1/2\end{bmatrix}_D$$

$c_1=0,\ c_2=-1/4,\ c_3=1/2$

- $\vec{e}_2=(-1/8)\vec{\beta}_1+(1/4)\vec{\beta}_2$

$$f(\vec{e}_2)=(-1/8)f(\vec{\beta}_1)+(1/4)f(\vec{\beta}_2)$$

$$=(-1/8)\begin{bmatrix}1\\1\\1\end{bmatrix}+(1/4)\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$f(\vec{e}_2)=\begin{bmatrix}1/8\\3/8\\-1/8\end{bmatrix}\longrightarrow\begin{bmatrix}1/8\\3/8\\-1/8\end{bmatrix}=c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\-2\\0\end{bmatrix}+c_2\begin{bmatrix}1\\0\\1\end{bmatrix}\longleftarrow \mathrm{Rep}_D(f(\vec{e}_1))=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}_D$$

$$\mathrm{Rep}_e(f(\vec{e}_2))=\begin{bmatrix}1/8\\3/8\\-1/8\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&0&1&1/8\\0&-2&0&3/8\\0&0&1&-1/8\end{array}\right]\xrightarrow{RREF}\begin{array}{cccc}c_1&c_2&c_3&\\ \end{array}\left[\begin{array}{ccc|c}1&0&0&1/4\\0&1&0&-3/16\\0&0&1&-1/8\end{array}\right]\longrightarrow \mathrm{Rep}_D(f(\vec{e}_2))=\begin{bmatrix}1/4\\-3/16\\-1/8\end{bmatrix}_D$$

$c_1=1/4,\ c_2=-3/16,\ c_3=-1/8$

— Here, $\mathrm{Rep}_{B,D}(f)=\begin{bmatrix}|&|\\ \mathrm{Rep}_D(f(\vec{e}_1))&\mathrm{Rep}_D(f(\vec{e}_2))\\ |&|\end{bmatrix}_{B,D}=\begin{bmatrix}0&1/4\\-1/4&-3/16\\1/2&-1/8\end{bmatrix}_{B,D}$

so that $\mathrm{Rep}_{B,D}(f)\,\mathrm{Rep}_B(\vec{v})=\mathrm{Rep}_D(f(\vec{v}))$

$$\begin{bmatrix}0&1/4\\-1/4&-3/16\\1/2&-1/8\end{bmatrix}_{B,D}\begin{bmatrix}c_1\\c_2\end{bmatrix}_B=\begin{bmatrix}(1/4)c_2\\(-1/4)c_1+(-3/16)c_2\\(1/2)c_1+(-1/8)c_2\end{bmatrix}_D\longleftarrow \text{"}A\vec{x}=\vec{b}\text{"}$$

- check: $\vec{v}=\begin{bmatrix}1/2\\0\end{bmatrix}_B=\mathrm{Rep}_B(\vec{e}_1)$; $\begin{bmatrix}0&1/4\\-1/4&-3/16\\1/2&-1/8\end{bmatrix}_{B,D}\begin{bmatrix}1/2\\0\end{bmatrix}_B=\begin{bmatrix}0\\(-1/4)(1/2)+(-3/16)(0)\\(1/2)(1/2)+(-1/8)(0)\end{bmatrix}_D=\begin{bmatrix}0\\-1/8\\1/4\end{bmatrix}_D=\mathrm{Rep}_D(f(\vec{e}_1))$ ✓

— Using the standard bases $\mathrm{Rep}_e(f)=\begin{bmatrix}|&|\\f(\vec{e}_1)&f(\vec{e}_2)\\|&|\end{bmatrix}_e=\begin{bmatrix}1/2&1/8\\1/2&3/8\\1/2&-1/8\end{bmatrix}_e$

so that $\mathrm{Rep}_e(f)\,\mathrm{Rep}_e(\vec{v})=\mathrm{Rep}_e(f(\vec{v}))$

$$\begin{bmatrix}1/2&1/8\\1/2&3/8\\1/2&-1/8\end{bmatrix}_e\begin{bmatrix}c_1\\c_2\end{bmatrix}_e=\begin{bmatrix}(1/2)c_1+(1/8)c_2\\(1/2)c_1+(3/8)c_2\\(1/2)c_1+(-1/8)c_2\end{bmatrix}_e\longleftarrow \text{"}A\vec{x}=\vec{b}\text{"}$$

- check: $\vec{v}=\begin{bmatrix}2\\0\end{bmatrix}_e=\mathrm{Rep}_e(\vec{v})$; $\begin{bmatrix}1/2&1/8\\1/2&3/8\\1/2&-1/8\end{bmatrix}_e\begin{bmatrix}2\\0\end{bmatrix}_e=\begin{bmatrix}(1/2)2+(1/8)0\\(1/2)2+(3/8)0\\(1/2)2+(-1/8)0\end{bmatrix}_e=\begin{bmatrix}1\\1\\1\end{bmatrix}_e$ ✓

Matrix Representations of Linear Transformations

* Let $f: V\rightarrow W$ be a linear map (homomorphism).
  For any $\vec{v}=c_1\vec{\beta}_1+\dots+c_n\vec{\beta}_n$ with respect to basis $B=\{\vec{\beta}_1,\dots,\vec{\beta}_n\}$,
  we have $f(\vec{v})=c_1f(\vec{\beta}_1)+\dots+c_nf(\vec{\beta}_n)$ with respect to basis $D=\{f(\vec{\beta}_1),\dots,f(\vec{\beta}_n)\}$.

Given vector spaces V with basis B and W with basis D for $f: V\rightarrow W$, the matrix representation of f is

$$Rep_{B,D}(f) = \begin{bmatrix} | & & | \\ Rep_D(f(\vec{\beta}_1)) & \cdots & Rep_D(f(\vec{\beta}_n)) \\ | & & | \end{bmatrix}.$$

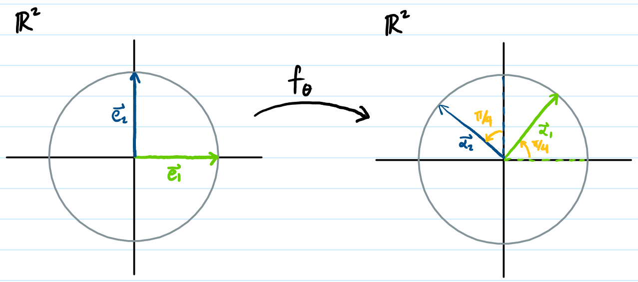
Each column of $Rep_{B,D}(f)$ is the coordinate vector of $f(\vec{\beta}_i)$ for $i = 1, \ldots, n$ in W.

If $A = Rep_{B,D}(f)$ is the matrix representation of $f: V \to W$, then

$$A\, Rep_B(\vec{v}) = Rep_D(f(\vec{v})). \longleftarrow \text{"}A\vec{x} = \vec{b}\text{"}$$

Rotation Transformations

Let $f_{\pi/4}: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map that rotates vectors in $\mathbb{R}^2$ into $\mathbb{R}^2$ counterclockwise by $\pi/4$.

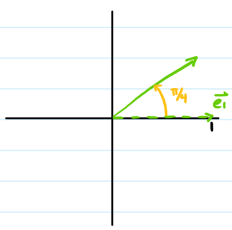


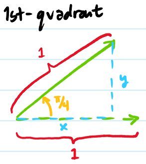
* Example 1: Suppose we have basis vectors $B = \{\vec{e}_1, \vec{e}_2\}$ in $\mathbb{R}^2$.

- Here, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- We rotate $\vec{e}_1$ and $\vec{e}_2$ by $\theta = \pi/4$.

  → Define action for $\vec{e}_1$:



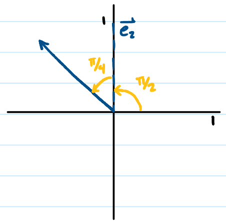


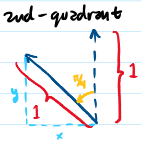
$$\cos(0) = \frac{x}{1} \longrightarrow x = \cos(0) \xrightarrow{\pi/4} x = \cos(0 + \pi/4) = 1/\sqrt{2}$$

$$\sin(0) = \frac{y}{1} \longrightarrow y = \sin(0) \xrightarrow{\pi/4} y = \sin(0 + \pi/4) = 1/\sqrt{2}$$

$$\text{So, } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{f_{\pi/4}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

  → Define action for $\vec{e}_2$:





$$\cos(\pi/2) = \frac{x}{1} \longrightarrow x = \cos(\pi/2) \xrightarrow{\pi/4} x = \cos(\pi/2 + \pi/4) = -1/\sqrt{2}$$

$$\sin(\pi/2) = \frac{y}{1} \longrightarrow y = \sin(\pi/2) \xrightarrow{\pi/4} y = \sin(\pi/2 + \pi/4) = 1/\sqrt{2}$$

$$\text{So, } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{f_{\pi/4}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

- The matrix representation of f is $Rep_{e,e}(f) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ with respect to the standard basis vectors.

  Here, we started with transforming the standard basis vectors. So, the matrix transformation is nonsingular.

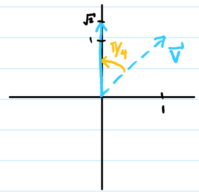
  So, for any vector $\vec{v}$, $f_{\pi/4}: \mathbb{R}^2 \to \mathbb{R}^2$ is $Rep_{e,e}(f) Rep_e(\vec{v}) = Rep_e(f(\vec{v}))$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \vec{v} = \vec{b}.$$

check: $\vec{v} = \begin{bmatrix}1\\0\end{bmatrix} \longrightarrow \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ ✓

$\vec{v} = \begin{bmatrix}0\\1\end{bmatrix} \longrightarrow \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ ✓

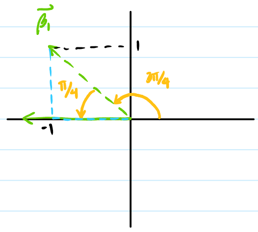
Try: $\vec{v} = \begin{bmatrix}1\\1\end{bmatrix} \longrightarrow \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} - 1/\sqrt{2}\\ 1/\sqrt{2} + 1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}0\\\sqrt{2}\end{bmatrix}$



* Example 2: Suppose we have basis vectors $B = \left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}\right\}$.

- Let $\vec{\beta}_1 = \begin{bmatrix}-1\\1\end{bmatrix}$ and $\vec{\beta}_2 = \begin{bmatrix}-1\\-1\end{bmatrix}$.

- We rotate $\vec{\beta}_1$ and $\vec{\beta}_2$ by $\pi/4$.



$\longrightarrow$ Define action for $\vec{\beta}_1$, length $\sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

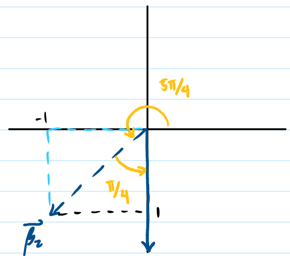
$$\begin{bmatrix}-1\\1\end{bmatrix} \xrightarrow{\pi/4} \begin{bmatrix}\sqrt{2}\cos(3\pi/4 + \pi/4)\\ \sqrt{2}\sin(3\pi/4 + \pi/4)\end{bmatrix} = \begin{bmatrix}-\sqrt{2}\\0\end{bmatrix}$$

2nd quadrant

$\longrightarrow$ Define action for $\vec{\beta}_2$, length $\sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$$\begin{bmatrix}-1\\-1\end{bmatrix} \xrightarrow{\pi/4} \begin{bmatrix}\sqrt{2}\cos(5\pi/4 + \pi/4)\\ \sqrt{2}\sin(5\pi/4 + \pi/4)\end{bmatrix} = \begin{bmatrix}0\\-\sqrt{2}\end{bmatrix}$$

3rd quadrant



- So, we have a set of action

$$\begin{bmatrix}-1\\1\end{bmatrix} \longrightarrow \begin{bmatrix}-\sqrt{2}\\0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}-1\\-1\end{bmatrix} \longrightarrow \begin{bmatrix}0\\-\sqrt{2}\end{bmatrix}$$

- For $f_{\pi/4}: \mathbb{R}^2 \to \mathbb{R}^2$, we need to find $Rep_e(f)$ with respect to the st. basis vectors, given basis vectors $B = \left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}\right\}$ and $D = \left\{\begin{bmatrix}-\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}0\\-\sqrt{2}\end{bmatrix}\right\}$

$\longrightarrow$ Here, we use the standard basis vectors and do a basis change first.

- $\vec{e}_1 = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 \longleftarrow Rep_B(\vec{e}_1) = \begin{bmatrix}c_1\\c_2\end{bmatrix}_B$

$\begin{bmatrix}1\\0\end{bmatrix} = c_1\begin{bmatrix}-1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\-1\end{bmatrix}$

$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$

$\downarrow$

$$\left[\begin{array}{cc|c}-1 & -1 & 1\\1 & -1 & 0\end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c}\begin{array}{cc}c_1 & c_2\end{array}\\ \left[\begin{array}{cc|c}1 & 0 & -1/2\\0 & 1 & -1/2\end{array}\right]\end{array} \longrightarrow Rep_B(\vec{e}_1) = \begin{bmatrix}-1/2\\-1/2\end{bmatrix}_B$$

$c_1 = -1/2,\ c_2 = -1/2$

- $\vec{e}_2 = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 \longleftarrow Rep_B(\vec{e}_2) = \begin{bmatrix}c_1\\c_2\end{bmatrix}_B$

$\begin{bmatrix}0\\1\end{bmatrix} = c_1\begin{bmatrix}-1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\0\end{bmatrix}$

$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$

$\downarrow$

$$\left[\begin{array}{cc|c}-1 & -1 & 0\\1 & -1 & 1\end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c}\begin{array}{cc}c_1 & c_2\end{array}\\ \left[\begin{array}{cc|c}1 & 0 & 1/2\\0 & 1 & -1/2\end{array}\right]\end{array} \longrightarrow Rep_B(\vec{e}_2) = \begin{bmatrix}1/2\\-1/2\end{bmatrix}_B$$

$c_1 = 1/2,\ c_2 = -1/2$

$\longrightarrow$ Next, we use the action for the transformations.

- $\vec{e}_1 = (-\tfrac{1}{2})\vec{\beta}_1 + (-\tfrac{1}{2})\vec{\beta}_2 \longleftarrow \text{Rep}_B(\vec{e}_1) = \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix}_B$

  $f(\vec{e}_1) = (-\tfrac{1}{2})f(\vec{\beta}_1) + (-\tfrac{1}{2})f(\vec{\beta}_2)$

  $\downarrow$

  $= (-\tfrac{1}{2})\begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix} + (-\tfrac{1}{2})\begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix}$

  $f(\vec{e}_1) = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \longrightarrow \text{Rep}_e(f(\vec{e}_1)) = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}_e$

- $\vec{e}_2 = (\tfrac{1}{2})\vec{\beta}_1 + (-\tfrac{1}{2})\vec{\beta}_2 \longleftarrow \text{Rep}_B(\vec{e}_2) = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}_B$

  $f(\vec{e}_2) = (\tfrac{1}{2})f(\vec{\beta}_1) + (-\tfrac{1}{2})f(\vec{\beta}_2)$

  $\downarrow$

  $= (\tfrac{1}{2})\begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix} + (-\tfrac{1}{2})\begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix}$

  $f(\vec{e}_1) = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \longrightarrow \text{Rep}_e(f(\vec{e}_2)) = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

$\longrightarrow$ Now, we can determine $\text{Rep}_e(f) = \begin{bmatrix} | & | \\ f(\vec{e}_1) & f(\vec{e}_2) \\ | & | \end{bmatrix}$

$$\text{Rep}_e(f) = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}_e$$

So, for any vector $\vec{v}$, the transformation $f_{\pi/4}: \mathbb{R}^2 \to \mathbb{R}^2$ is

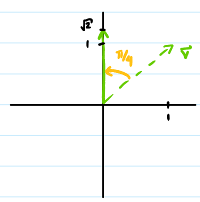
$$\text{Rep}_e(f)\,\text{Rep}_e(\vec{v}) = \text{Rep}_e(f(\vec{v}))$$

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\vec{v} = \vec{b}$$

Check: $\vec{\beta}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 - \sqrt{2}/2 \\ -\sqrt{2}/2 + \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}$ ✓

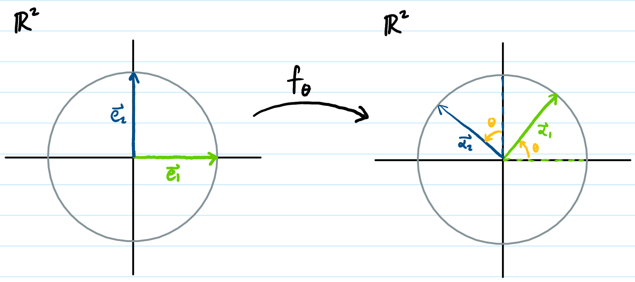
$\vec{\beta}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \longrightarrow \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 + \sqrt{2}/2 \\ -\sqrt{2}/2 - \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix}$ ✓

Try: $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 - \sqrt{2}/2 \\ \sqrt{2}/2 + \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$



2D General Rotation Transformations

Let $f_\theta: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map that rotates vectors in $\mathbb{R}^2$ into $\mathbb{R}^2$ counterclockwise by θ.



Consider a vector in $\mathbb{R}^2$, $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$.

- Represent $\vec{v}$ in terms sin and cos.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} L\cos(\alpha) \\ L\sin(\alpha) \end{bmatrix},$$ where α is the initial angle relative to x-axis, and for any vector lenght L.

- Then, rotate $\vec{v}$ by θ counter clockwise.

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} L\cos(\alpha+\theta) \\ L\sin(\alpha+\theta) \end{bmatrix}$$

- Using the angle sum rule

$$\begin{bmatrix} L\cos(\alpha+\theta) \\ L\sin(\alpha+\theta) \end{bmatrix} = \begin{bmatrix} L(\cos(\alpha)\cos(\theta) - \sin(\alpha)\sin(\theta)) \\ L(\sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta)) \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} Lx_1\cos(\theta) - Lx_2\sin(\theta) \\ Lx_1\sin(\theta) + Lx_2\cos(\theta) \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{f_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \text{ matrix representation}} \begin{bmatrix} Lx_1 \\ Lx_2 \end{bmatrix}$$

- Given basis vectors $\vec{e_1}$ and $\vec{e_2}$, the action for the general 2D rotation for a angle θ is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}.$$

The matrix representation for $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$Rep_e(f_\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

So that for any vector $\vec{v}$ with respect to the standard basis,

$$Rep_e(f_\theta)\, Rep_e(\vec{v}) = Rep_e(f_\theta(\vec{v}))$$

or

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \vec{v} = \vec{b}.$$

- The 2D rotation map $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix representation $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ can be shown to be homomorphic (linear map) and bijective (one-to-one and onto).

$\longrightarrow$ homomorphism: Let $\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ and scalars $c_1, c_2 \in \mathbb{R}$.

$$f(c_1\vec{v_1} + c_2\vec{v_2}) = f\left(c_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)(c_1x_1 + c_2x_2) - \sin(\theta)(c_1y_1 + c_2y_2) \\ \sin(\theta)(c_1x_1 + c_2x_2) + \cos(\theta)(c_1y_1 + c_2y_2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)c_1x_1 + \cos(\theta)c_2x_2 - \sin(\theta)c_1y_1 - \sin(\theta)c_2y_2 \\ \sin(\theta)c_1x_1 + \sin(\theta)c_2x_2 + \cos(\theta)c_1y_1 + \cos(\theta)c_2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)c_1x_1 - \sin(\theta)c_1y_1 + \cos(\theta)c_2x_2 - \sin(\theta)c_2y_2 \\ \sin(\theta)c_1x_1 + \cos(\theta)c_1y_1 + \sin(\theta)c_2x_2 + \cos(\theta)c_2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)c_1x_1 - \sin(\theta)c_1y_1 \\ \sin(\theta)c_1x_1 + \cos(\theta)c_1y_1 \end{bmatrix} + \begin{bmatrix} \cos(\theta)c_2x_2 - \sin(\theta)c_2y_2 \\ \sin(\theta)c_2x_2 + \cos(\theta)c_2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} c_1x_1 \\ c_1y_1 \end{bmatrix} + \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} c_2x_2 \\ c_2y_2 \end{bmatrix}$$

$$f(c_1\vec{v_1} + c_2\vec{v_2}) = f(c_1\vec{v_1}) + f(c_2\vec{v_2})$$

Thus, f is homomorphic.

→ Bijection:

- one-to-one;

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ for some vector } \vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{cc|c} \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

The only solution to the homogeneous equation is $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$ for any θ.

This means $\text{Null}\left(\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\right) = \{\vec{0}\}$ with dimension 0.

Thus, f is one-to-one.

- onto;

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ for some vectors } \vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \begin{bmatrix} a \\ b \end{bmatrix};\ a, b \in \mathbb{R}.$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} \cos(\theta) & -\sin(\theta) & a \\ \sin(\theta) & \cos(\theta) & b \end{array}\right]$$

– Case 1: If $\theta = \pi/2 + n\pi$ for $n = 1, 2, \ldots$, then $\left[\begin{array}{cc|c} 0 & \pm 1 & a \\ \pm 1 & 0 & b \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & \pm a \\ 0 & 1 & \pm b \end{array}\right]$

– Case 2: If $\theta \neq \pi/2 + n\pi$ for $n = 1, 2, \ldots$, then $\left[\begin{array}{cc|c} \cos(\theta) & -\sin(\theta) & a \\ \sin(\theta) & \cos(\theta) & b \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{cc|c} \cos(\theta) & -\sin(\theta) & a \\ 0 & 1 & -a\sin(\theta) + b\cos(\theta) \end{array}\right]$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & (a + b\sin(\theta)\cos(\theta) - a\sin^2(\theta))(1/\cos(\theta)) \\ 0 & 1 & -a\sin(\theta) + b\cos(\theta) \end{array}\right]$$

So, there is at least one solution $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ for any $\begin{bmatrix} a \\ b \end{bmatrix}$ and θ. If $\cos(\theta) = 0$

This means $\text{Col}\left(\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}\right\}$ with dimension 2.

The rank of the matrix is 2 which is equal to the dimensions of $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix} \in \text{Col}\left(\begin{bmatrix} \sin(\theta) & -\sin(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}\right)$.

Thus, f is onto.

Since $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ is homomorphic and bijective, then it is isomorphic.

Thus, f_θ is isomorphic.