Null Space and Column Space

Objectives:

1. Define the Column space
2. Define the Null Space
3. Discuss how the Column and null space are subspaces for solving linear systems.

Recall: Homomorphisms

A homomorphic map only need to preserve structure, meaning
- closed under addition and
- closed under scalar multiplication.

It does not to be injective (one-to-one) nor surjective (onto).

Example: Let $f: \mathbb{R}^2 \to \mathbb{R}^3$.

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{f} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$ is a homomorphism

and it is just onto, and not one-to-one.

$\downarrow$

$\begin{bmatrix} x \\ y \end{bmatrix}$ onto the xy-plane in $\mathbb{R}^3$.

Linear System of Equations

* In matrix-vector form,

$$\underset{m\times n}{A}\ \underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$$

where m is the number of rows (# of equations)
n is the number of columns (# of variables).

Here, vector $\vec{x} \in \mathbb{R}^n$ is the input and $\vec{b} \in \mathbb{R}^m$ is the output.
The matrix A is a linear operator that transforms $\vec{x}$ into $\vec{b}$.
More formally,

$$f: \mathbb{R}^n \to \mathbb{R}^m$$

where $\mathbb{R}^n$ and $\mathbb{R}^m$ are vector spaces of dimension m and n respectively and f is the mapping represented by a matrix.

* $A\vec{x} = \vec{b}$ is a homomorphism because m and n does not have to be equal or it does not have to be one-to-one or onto.

* Under a homomorphism, the image (column space) of any subspace of the domain $(\mathbb{R}^n)$ is a subspace of the codomain $(\mathbb{R}^m)$.

* The column space of the entire $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^m$.

<u>The Column Space</u>

Let $f: V \to W$ be a homomorphism.

* The <u>column space</u> (also called the <u>range</u> or <u>image</u>) of f is the set of all vectors in W than can be written as $f(\vec{v})$ for some $\vec{v} \in V$:

$$Col(f) = \{\vec{w} \in W \mid \vec{w} = f(\vec{v}) \text{ for some } \vec{v} \in V\}.$$

* If A is the matrix representing f with respect to some bases of V and W, the <u>column space</u> of A is the span of its columns:

$$Col(A) = Span\{\text{columns of } A\}.$$

Example: Let $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & -1 \\ -6 & -2 & 0 \end{bmatrix}$.

Here, $f: V \to W$ is $A: \mathbb{R}^3 \to \mathbb{R}^3$ since A is a 3x3 matrix.

$$\begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & -1 \\ -6 & -2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Columns with pivots

$$Col(A) = Span\left\{\begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\} \longleftarrow \dim(Col(A)) = 2$$

rank(A) = 2

or

or

$$Col(A) = \left\{ c_1 \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}$$

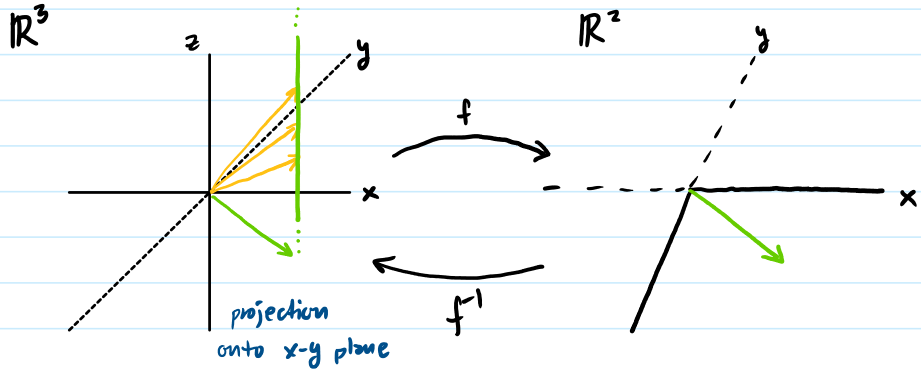
So, in order for $A\vec{x} = \vec{b}$ to have at least one solution, $\vec{b} \in \mathbb{R}^m$ must be in the $Col(A)$ because $Col(A)$ is a subspace of $\mathbb{R}^m$.

Inverse Images

Examples:

1. Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{f} \begin{bmatrix} x \\ y \end{bmatrix}$

   be a homomorphism that is many-to-one.
   This example is a projection from $\mathbb{R}^3$ to $\mathbb{R}^2$.

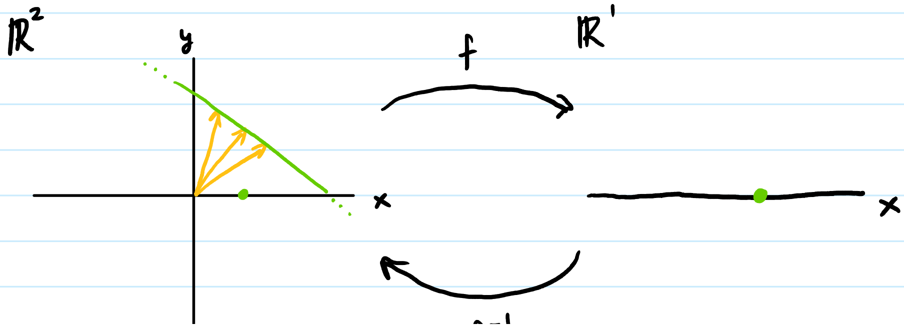


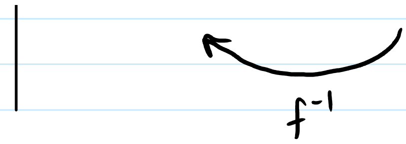
So, $f^{-1}: \mathbb{R}^2 \to \mathbb{R}^3$.

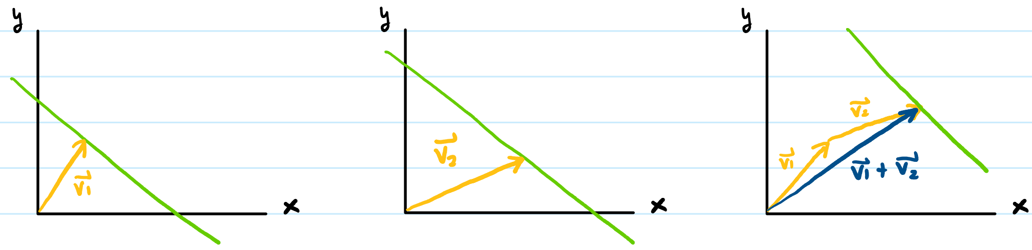
2. Let $f: \mathbb{R}^2 \to \mathbb{R}^1$ be a homomorphism defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{f} x + y.$$

This is an example of projection a vector onto a scalar (many-to-one).







The sum of the images is the image of the sum.
That is $f(\vec{v}_1) + f(\vec{v}_2) = f(\vec{v}_1 + \vec{v}_2)$.

* For any homomorphism, the inverse image of a subspace of the range is a subspace of the domain.
The inverse image of the trivial subspace $(\vec{0})$ of the range is a subspace of the domain.

Let $f: V \to W$ be a homomorphism.
So, the $Col(f) = \{\vec{w} \in W \mid \vec{w} = f(\vec{v}) \text{ for } \vec{v} \in V\}$

and the inverse image is $f^{-1}(\vec{w}) = \{\vec{v} \in V \mid f(\vec{v}) = \vec{w} \text{ for } \vec{w} \in W\}$.

Column Space and Linear Systems

Let $A\vec{x} = \vec{b}$ be a homomorphism $f: \mathbb{R}^n \to \mathbb{R}^m$ where f is represented as A.

* Here $\vec{x} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$.

* If $\dim(Col(A)) = m$ (dimension of $Col(A)$ is equal to the number of rows),
$\downarrow$
matrix rank is m (full rank)
then A is surjective (onto).
Also A, has an inverse, A^{-1}, so that $\vec{x} = A^{-1}\vec{b}$.

* If $\dim(Col(A)) < m$ (dimension of $Col(A)$ is less than the number of rows),
$\downarrow$
matrix rank is $< m$ (low rank)
then A is still surjective (onto) because $\vec{b} \in \mathbb{R}^m$ can still be in the $Col(A)$.
However A does not have an inverse A^{-1}

Then A is still surjective (onto) because $\vec{b} \in \mathbb{R}^m$ can still be in the $Col(A)$.
However, A does not have an inverse A^{-1}.

* If $A\vec{x} = \vec{b}$ has no solution, meaning $\vec{b} \in \mathbb{R}^m$ is not an element of $Col(A)$, then A is not onto.

The Null Space

Let $f : V \to W$ be a homomorphism.

* The <u>null space</u> (also called the kernel) of f is the set of all vectors in V that map to the zero vector in W:

$$Null(f) = \{\vec{x} \in V \mid f(\vec{x}) = \vec{0}\}.$$

* If A is the matrix representing f with respect to some bases V and W, the null space of A is the set of all solutions to the homogeneous system:

$$Null(A) = \{\vec{x} \in V \mid A\vec{x} = \vec{0}\}.$$

* In relation to inverse images:
  - $\to$ Let S be a subspace of the column space of f. Consider the inverse image of S. It is nonempty because it contains $\vec{0}_V$, since $f(\vec{0}_V) = \vec{0}_W$ is an element of S as S is a subspace.
  - $\to$ Let $\vec{v}_1, \vec{v}_2 \in V$ and $f(\vec{v}_1), f(\vec{v}_2) \in W$ so that they are elements of S. Then, $c_1\vec{v}_1 + c_2\vec{v}_2$ is an element of $f^{-1}(S)$ because
  $$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2) \in S.$$
  - $\to$ In other words, the null space of a linear map $f : V \to W$ is the inverse image of $\vec{0}_W$.
  $$Null(f) = f^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid f(\vec{v}) = \vec{0}_W\}.$$

* $\dim(Null(A))$ is called the nullity.

Example: Let $A\vec{x} = \vec{0}$ where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & -1 \\ -6 & -2 & 0 \end{bmatrix}.$$

Here, $A: \mathbb{R}^3 \to \mathbb{R}^3$ since A is a 3x3 matrix and $\vec{0}_{\mathbb{R}^3}$. ← the $\vec{0}$ under $\mathbb{R}^3$.

$$\begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & -1 \\ -6 & -2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & \boxed{1/3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

↳ free variable

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, let $x_2 = s$.

↓

free variable

$$\left.\begin{aligned} x_1 + \tfrac{1}{3}s &= 0 \\ x_2 &= s \\ x_3 &= 0 \end{aligned}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$$

So,

$$\text{Null}(A) = \text{Span}\left\{ \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

← $\dim(\text{Null}(A)) = 1$ nullity of 1.

or

$$\text{Null}(A) = \left\{ s \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}, s \in \mathbb{R} \right\}$$

The Null Space and Linear Systems

Let $A\vec{x} = \vec{0}$ be a homomorphism $f: \mathbb{R}^n \to \vec{0}_{\mathbb{R}^m}$ where f is represented as A.

* Here, $\vec{x} \in \mathbb{R}^n$ and $\vec{0}$ lives in $\mathbb{R}^m$.

* If $\dim(\text{Null}(A)) = 0$,

  ↓

  the trivial solution

  then the only solution to $A\vec{x} = \vec{0}$ is $\vec{0}_{\mathbb{R}^n}$.

  This means that A is one-to-one.

* If $\dim(\text{Null}(A)) \neq 0$,

  ↓

  infinite solution

  then there is at least one solution to $A\vec{x} = \vec{0}$.

  This means that A is not one-to-one.

The rank-nullity Theorem

* A linear map's rank and its nullity equals the dimension of its domain.

* In other words, $\dim(V) = \dim(\text{Null}(A)) + \dim(\text{Col}(A))$
for $f: V \to W$ where f is represented as A.

Examples:

- Let $A\vec{x} = \vec{b}$ where
$$A = \begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & -1 \\ -6 & -2 & 0 \end{bmatrix}.$$

Here, $f: \mathbb{R}^3 \to \mathbb{R}^3$ where f is A, $\vec{x} \in \mathbb{R}^3$, $\vec{b} \in \mathbb{R}^3$

- $\dim(\mathbb{R}^3) = 3$
- $\dim(\text{Null}(A)) = 1 \longleftarrow$ nullity
- $\dim(\text{Col}(A)) = 2 \longleftarrow$ rank

$\left.\right\}$ $3 = 1 + 2$
$3 = \dim(\mathbb{R}^3)$

Also, A is onto since $\dim(\text{Col}(A)) = 2$, meaning every element $\vec{x} \in \mathbb{R}^n$ is mapped to $\mathbb{R}^m$.
In addition, A is not one-to-one since $\dim(\text{Null}(A)) = 1$, meaning there is more than one $\vec{x} \in \mathbb{R}^n$ mapped to $\vec{0}_{\mathbb{R}^m}$.

- Let $f: \mathbb{R}^3 \to \mathbb{R}^4$ be a homomorphism defined by
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{f} \begin{bmatrix} x \\ 0 \\ y \\ 0 \end{bmatrix}$$

The $\text{Col}(f) = \left\{ \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$ and $\text{Null}(f) = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \middle| \, z \in \mathbb{R} \right\}$

two variables

one variable

$\dim(\text{Col}(f)) = 2$
(rank of f)

$\dim(\text{Null}(f)) = 1$
(nullity of f)

So, $\dim(\text{Col}(f)) + \dim(\text{Null}(f)) = 2 + 1 = 3 \longrightarrow \dim(\mathbb{R}^3)$.

* The rank of a linear map is less than or equal to the dimension of the domain. Equality holds if and only if the nullity of the map is 0.

In other words, if $A\vec{x} = \vec{b}$ is a homomorphism $f: \mathbb{R}^n \to \mathbb{R}^m$, then

$$\text{rank}(A) \leq \dim(\mathbb{R}^n)$$
and

$$\mathrm{rank}(A) \le \dim(\mathbb{R}^n)$$

and

$$\mathrm{rank}(A) \le \dim(\mathrm{Null}(A)) + \dim(\mathrm{Col}(A))$$

$\longrightarrow$ $\mathrm{rank}(A) = \dim(\mathrm{Col}(A))$ if $\dim(\mathrm{Null}(A)) = 0$.

$\longrightarrow$ $\dim(\mathrm{Null}(A)) = 0$ if $\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = \dim(\mathbb{R}^n)$.

Equivalent Statements

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ and f is represented as a matrix A, so that $A\vec{x} = \vec{b}$ where $\vec{x} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^n$. Here, number of equations is equal to number of variables.

1. A is one-to-one.
2. A has an inverse, A^{-1}, so that $\vec{x} = A^{-1}\vec{b}$.
3. $\mathrm{Null}(A) = \{\vec{0}\}$, that is $\dim(\mathrm{Null}(A)) = 0$.
4. $\mathrm{rank}(A) = n$ or $\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = \dim(\mathbb{R}^n)$.
5. If $\{\vec{\beta}_1, \dots, \vec{\beta}_n\}$ is a basis for $\mathbb{R}^n$, then $\{f(\vec{\beta}_1), \dots, f(\vec{\beta}_n)\}$ is a basis for $\mathbb{R}^n$.

$\rightarrow$ these describes a unique solution to $A\vec{x} = \vec{b}$, where A is square.