Solution Sets

Objectives:

1. Define solution sets
2. Layout types of solutions
3. Introduce the solutions in parametric form

Example 1: (unique solution)

$$\begin{aligned} 2x+y&=5 \\ x-y&=1 \end{aligned} \xrightarrow{\text{augmented matrix}} \begin{array}{c} \begin{array}{ccc} x & y & \text{"="} \end{array} \\ \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -1 & 1 \end{array}\right] \begin{array}{l} \to R_1 \\ \to R_2 \end{array} \end{array}$$

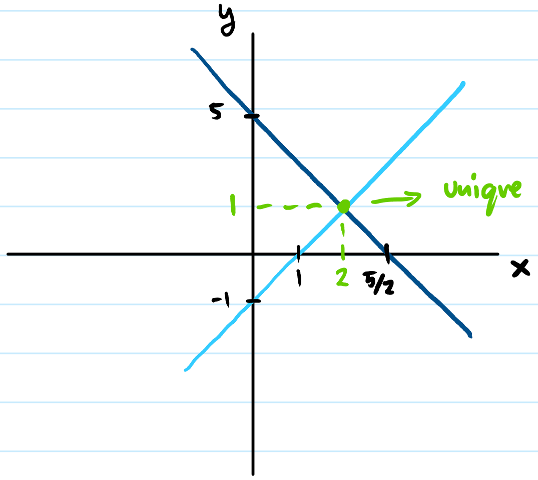
$$\xrightarrow{\text{switch } R_1 \,\&\, R_2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{R_2 = -2R_1 + R_2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 = (1/3)R_2} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 1 \end{array}\right] \end{array} \quad \text{row echelon form}$$

$$\begin{aligned} x-y&=1 \\ y&=1 \end{aligned} \longrightarrow \left.\begin{aligned} x&=2 \\ y&=1 \end{aligned}\right\} \text{consistent}$$

* unique solution



unique solution $(x,y) = (2,1)$

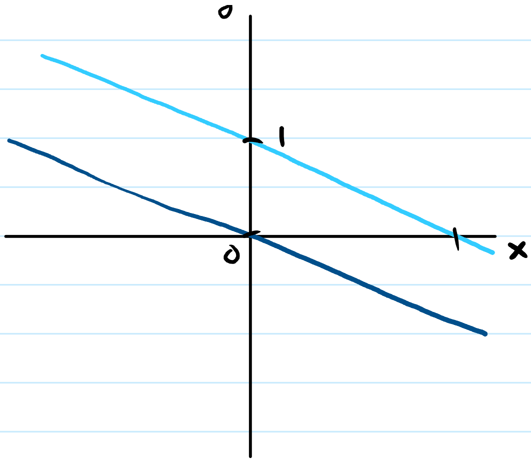
Example 2: (no solution)

$$\begin{aligned} 2x+y&=0 \\ 4x+2y&=2 \end{aligned} \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & \text{"="} \end{array} \\ \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \end{array} \end{array}$$

$$\xrightarrow{R_1 = (1/2)R_1} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 4 & 2 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 = -4R_1 + R_2} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & \boxed{0} & \boxed{2} \end{array}\right] \end{array} \quad \text{row echelon form}$$

$0 = 2$ false or inconsistent



$\downarrow$

$0 = 2$ false or inconsistent

* no solution

parallel lines

Example 3: (infinite solutions)

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 6 \end{aligned} \longrightarrow \begin{array}{cc} x & y & & \\ \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 6 \end{array}\right] & \begin{array}{c} R_1 \\ R_2 \end{array} \end{array}$$

$$\xrightarrow{R_2 = -2R_1 + R_2} \begin{array}{cc} x & y \\ \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

row echelon form

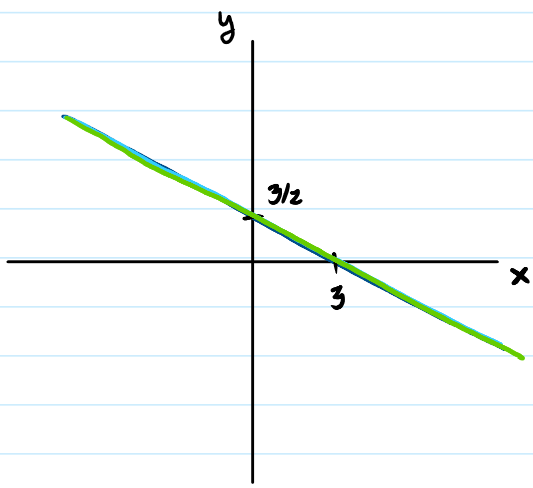
$\rightarrow$ zero row

$x + 2y = 3$

Then solve for y:

$y = \frac{3}{2} - \frac{x}{2} \longrightarrow$ consisent

* infinite solutions



Both lines are intersecting everywhere.

* solutions in parametric form

$$\begin{array}{cc} x & y \\ \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right] \end{array} \longrightarrow x + 2y = 3$$

$\downarrow$

y is a free variable

Let $y = s$. So $x + 2y = 3 \rightarrow x + 2s = 3$

$x = 3 - 2s$

Arrangement: $x = 3 - 2s$ for $s \in (-\infty, \infty)$.

Arrangement: $\begin{aligned} x &= 3 - 2s \\ y &= s \end{aligned}$ for $s \in (-\infty, \infty)$.

$\downarrow$ or

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} s \longrightarrow \text{vector form}$$

Definition (solution sets): Consider a system of m linear equations with n variables $x_1, x_2, \cdots, x_n$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where a_{ij} and b_i are constants.

The solution set is: $\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid \text{all } m \text{ equations are satisfied}\}$.

Cases: 1. If there is exactly one solution, then the set is a single point in n-dimensional space;
$\{(x_1, x_2, \cdots, x_n)\}$ (a single poin in $\mathbb{R}^n$).

2. If there are infinitely many solution, then the set forms a line, plane, or higher-dimensional flat affine subspace;
$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i = \text{expression in free parameters})$
(a line, plane, or high-dimensional flat).

3. If there is no solution, then the set is empty;
$\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n\} = \phi$ or $\{\}$