Subspace and Span

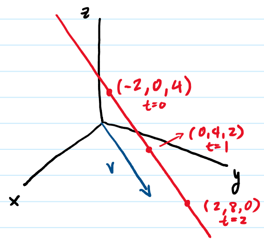
Objectives
1. Parametric equations for lines and planes
2. Define the spanning set
3. Define basis vectors

Parametric Equations for a line in space

The line parallel to $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ passing through point (x_0, y_0, z_0)

is $\left.\begin{array}{l} x = x_0 + t v_1 \\ y = y_0 + t v_2 \\ z = z_0 + t v_3 \end{array}\right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad -\infty < t < \infty.$

⟶ Example:
Find the line parallel to $\vec{v} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ through the point $(-2, 0, 4)$.



$$\left.\begin{array}{l} x = -2 + 2t \\ y = 0 + 4t \\ z = 4 - 2t \end{array}\right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}}_{t=0} + t \boxed{\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}}$$

⟶ basis vector

⟶ Span $\left\{ \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \right\}$

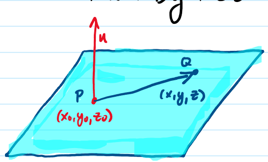
An Equation for a Plane in Space

The plane perpendicular to $\vec{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ passing through the point (x_0, y_0, z_0)

is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

↓ Simplified

$Ax + By + Cz = D$ where $D = Ax_0 + By_0 + Cz_0 = \vec{n} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$



$\vec{n} \cdot \vec{PQ} = 0$
$\vec{n} \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$
$\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$
$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$\vec{n} \cdot \vec{PQ} = 0$

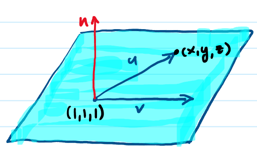
$\vec{n} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right) = 0$

$\vec{n} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$

Parametric Equations for a Plane in Space

The plane parallel to $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ passing through the point (x_0, y_0, z_0)

is $\left.\begin{array}{l} x = t u_1 + s v_1 + x_0 \\ y = t u_2 + s v_2 + y_0 \\ z = t u_3 + s v_3 + z_0 \end{array}\right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + s \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + t \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \begin{array}{l} -\infty < t < \infty \\ -\infty < s < \infty \end{array}$



Example:
Find the plane parallel to the vectors $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ passing through the point $(1, 1, 1)$.

$$\left.\begin{array}{l} x = t - s + 1 \\ y = t + s + 1 \\ z = -t + s + 1 \end{array}\right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{s=0,\ t=0} + \boxed{s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$$

basis vectors

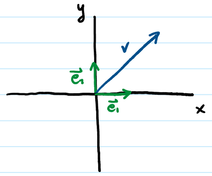
$$\left.\begin{array}{l} y = t + s + 1 \\ z = -t + s + 1 \end{array}\right\} \quad \begin{bmatrix} y \\ z \end{bmatrix} = \underset{s=0,\ t=0}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + s\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\longrightarrow \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \quad \text{basis vectors}$$

Basis Vectors

* Basis vectors is a set of independent vectors which span a vector space.

⟶ In $\mathbb{R}^2$



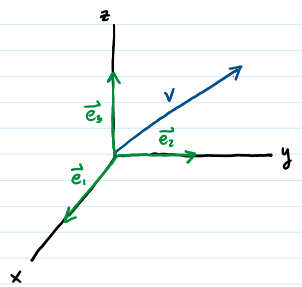
Basis vectors: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{set}} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Vectors using $\vec{e_1}$ & $\vec{e_2}$ basis: $\vec{v} = v_1\vec{e_1} + v_2\vec{e_2}$

$$= v_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

⟶ In $\mathbb{R}^3$



Basis Vectors: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\text{set}} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Vectors using the basis: $\vec{v} = v_1\vec{e_1} + v_2\vec{e_2} + v_3\vec{e_3}$

$$\vec{v} = v_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Basis Definition

* A basis has two conditions:

1. The basis set must be linearly independent.
2. The basis set must span the vector space.

* Linear Independence – Basic definition

A sequence of vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ is linearly independent if and only if the only scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\sum_{i=1}^{n} \lambda_i \vec{u}_i = 0$ are $\lambda_1 = \cdots = \lambda_n = 0$.

In other words, $\lambda_1\vec{u}_1 + \lambda_2\vec{u}_2 + \cdots + \lambda_n\vec{u}_n = 0$ is true only if $\lambda_1 = \cdots = \lambda_n = 0$.

If any $\lambda_1, \lambda_2, \dots, \lambda_n$ is non-zero, then it is linearly dependent.

⟶ Example: Let $\vec{u} = \underset{\vec{e_1}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$, $\vec{v} = \underset{\vec{e_2}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- Is $\vec{u}$ & $\vec{v}$ linearly independent?

$$\lambda_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} \lambda_1(1) + \lambda_2(0) = 0 \\ \lambda_1(0) + \lambda_2(1) = 0 \end{array} \longrightarrow \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 0 \end{array} \quad \checkmark \text{ yes.}$$

- Is $\vec{u}, \vec{v}$, & $\vec{w}$ linearly independent?

$$\lambda_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \lambda_3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} \lambda_1(1) + \lambda_2(0) + \lambda_3(1) = 0 \\ \lambda_1(0) + \lambda_2(1) + \lambda_3(1) = 0 \end{array}$$

λ_3 is a free variable

$$\begin{array}{l} \lambda_1 + + \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \end{array} \longrightarrow \begin{array}{c} \begin{array}{ccc|c} \lambda_1 & \lambda_2 & \lambda_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & \boxed{1} & 0 \\ 0 & 1 & \boxed{1} & 0 \end{array}\right] \end{array} \quad \times \text{ no.}$$

⟶ Example: Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$.

$\lambda_2 + \lambda_3 = 0 \qquad \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \end{array}\right]$

$\longrightarrow$ Example: Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$.

- Is $\vec{u}$ & $\vec{v}$ linearly independent?

$$\lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{cc} \lambda_1 & \lambda_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \end{array} \quad \checkmark \text{ yes.}$$

$\hookrightarrow \lambda_1 = 0,\ \lambda_2 = 0$

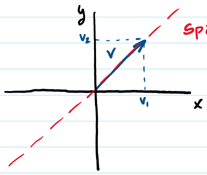
Span

$\ast$ If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are elements of a vector space V, then $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a subspace of V.

- Graphical Representation of span

$\longrightarrow$ Span of one vector in 2D

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

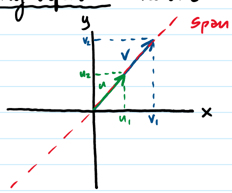


$\text{span}(v) \rightarrow \underbrace{\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\text{parameterized line in 2D}} \longrightarrow \begin{array}{l} x = tv_1 \\ y = tv_2 \end{array}$

$\longrightarrow$ Span of two linearly dependent vectors in 2D:

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

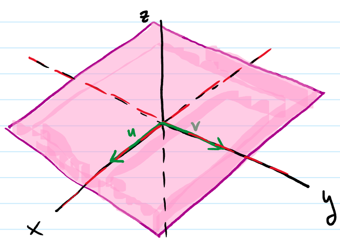


$\text{span}(u,v) \rightarrow \underbrace{\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\text{parameterized line in 2D}}$

$\longrightarrow$ Span of two linearly independent vectors in 3D:

$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

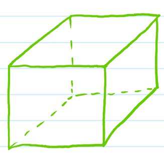
$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$



span(u,v) plane $\rightarrow \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{parameterized plane in 3D}} \longrightarrow \begin{array}{l} x = t \\ y = s \\ z = 0 \end{array}$

If we include $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, which is linearly independent to u and v, then span(u,v,w) is a cube.

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{parameterized cube in 3D}}$$



- Spanning sequence / spanning set is equivalent to system of linear equations.

$\longrightarrow$ Example: check if $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are a spanning sequence for $\mathbb{R}^2$.

$$\lambda \vec{u} + \beta \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{system of linear equations}} \left.\begin{array}{l} \lambda - \beta = x \\ \lambda + \beta = y \end{array}\right\} \begin{array}{l} \text{two equations} \\ \text{two unknowns} \end{array}$$

$\downarrow$ two lines in $\mathbb{R}^2$ with an intersection

$$\lambda = \frac{x+y}{2} \text{ and } \beta = \frac{y-x}{2}$$

$\lambda + \beta = y$ } two unknowns

$\downarrow$ two lines in $\mathbb{R}^2$ with an intersection

$\lambda = \frac{x+y}{2}$ and $\beta = \frac{y-x}{2}$

Check:

- $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} x=1 \\ y=1 \end{matrix} \rightarrow \lambda = 1, \beta = 0 \rightarrow \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{u}$ ✓
- $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} x=-1 \\ y=1 \end{matrix} \rightarrow \lambda = 0, \beta = 1 \rightarrow \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (1)\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{v}$ ✓

In the span of $\vec{u}$ & $\vec{v}$ {
- $\vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \begin{matrix} x=1 \\ y=-1 \end{matrix} \rightarrow \lambda = 0, \beta = -1 \rightarrow \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{w}$ ✓
- $\vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \rightarrow \begin{matrix} x=2 \\ y=3 \end{matrix} \rightarrow \lambda = \frac{5}{2}, \beta = \frac{1}{2} \rightarrow \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \vec{b}$ ✓

$\longrightarrow$ Example: check if $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are spanning sequence for $\mathbb{R}^3$.

$$\lambda \vec{v}_1 + \beta \vec{v}_2 + \gamma \vec{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\lambda \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{\text{system of linear equations}} \begin{matrix} \lambda + \gamma = x \\ -\lambda + \beta = y \\ -\beta - \gamma = z \end{matrix}$$

} three equations three unknowns

$\downarrow$ three planes in $\mathbb{R}^3$ with an intersection

$\lambda = x - \gamma$

$-(x-\gamma) + \beta = y \rightarrow -x + \gamma + \beta = y \rightarrow \beta = y + x - \gamma$

$-(y+x-\gamma) - \gamma = z \rightarrow -y - x = z \rightarrow x+y+z=0.$

The system has solutions if $x+y+z=0$.
If $x+y+z \neq 0$, then it has no solution.

Check:

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rightarrow 1-1+0=0$ ✓

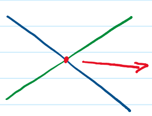
$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \rightarrow 0+1-1=0$ ✓

$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightarrow 1+0-1=0$ ✓

- $\vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in the span of V_1, V_2, and V_3 because $1+0+0=1 \neq 0$.
- $\vec{v}_5 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is in the span of V_1, V_2, and V_3 because $1+1-2=0$.

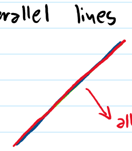
Points of Intersections

- Two lines intersecting



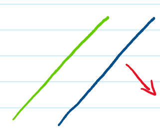
(unique solution) $\rightarrow$ A point of intersection of two lines.

- Parallel lines



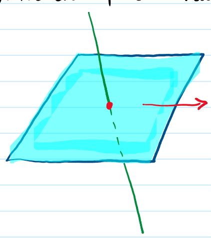
(infinite solutions) all points intersecting

or



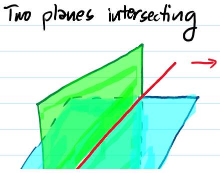
(no solution) no intersections

- A line and a plane intersecting



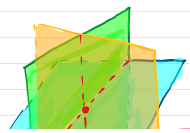
(unique solution) $\rightarrow$ A point of intersection on the plane

- Two planes intersecting

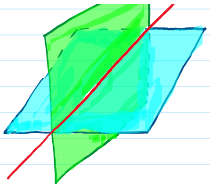


(infinite solutions) $\rightarrow$ line of intersection between two planes

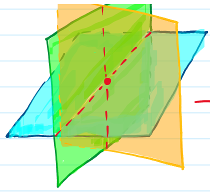
- Three planes intersecting



(unique solution)



between two planes



(unique solution) → A point of intersection of three planes.

⟶ Example:

Find vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

$$\left[\begin{array}{ccc|c} 3 & 6 & -2 & 15 \\ 2 & 1 & -2 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c|c} 1 & 0 & -10/9 & 5/3 \\ 0 & 1 & 2/9 & 5/3 \end{array}\right]$$

z is a free variable

Let $z = t$

Solution: $\left.\begin{array}{l} x - 10/9\, t = 5/3 \\ y + 2/9\, t = 5/3 \\ z = t \end{array}\right\}$ $\left.\begin{array}{l} x = 5/3 + 10/9\, t \\ y = 5/3 - 2/9\, t \\ z = t \end{array}\right\}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/3 \\ 0 \end{bmatrix} + t \boxed{\begin{bmatrix} 10/9 \\ -2/9 \\ 1 \end{bmatrix}} \longrightarrow \text{Span}\left\{\begin{bmatrix} 10/9 \\ -2/9 \\ 1 \end{bmatrix}\right\}$

↓

A line with basis vector $\begin{bmatrix} 10/9 \\ -2/9 \\ 1 \end{bmatrix}$

So, the solution set is $\left\{ \begin{bmatrix} 5/3 \\ 5/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 10/9 \\ -2/9 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$. The basis vector $\begin{bmatrix} 10/9 \\ -2/9 \\ 1 \end{bmatrix}$ is parallel to the line of Intersection.

The line of intersection does not pass through $\vec{0}$. So, S is <u>not</u> a subspace of $\mathbb{R}^3$.