Vector Space

Objectives:

1. Define the column space
2. Define the row space
3. Define matrix rank and how it relates to solutions of linear systems

Recall: System of Equations

$A\vec{x} = \vec{b}$, where A is an $m \times n$ matrix

$\vec{x}$ is a $n \times 1$ vector & (or $\vec{x} \in \mathbb{R}^n$)

$\vec{b}$ is a $m \times 1$ vector. (or $\vec{b} \in \mathbb{R}^m$)

Column Space (Notation and How to find them)

• Example:

$$A = \begin{bmatrix} 1 & 3 & -3 \\ -1 & -3 & 1 \\ 2 & 6 & -2 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 3 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

columns with pivots

linearly independent columns are columns 1 & 3 of A.

$$\text{col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} \right\}$$

$\dim(\text{col}(A)) = 2 \longleftarrow$ dimension of the column space

Row Space (Notation and How to find them)

• Example:

$$A = \begin{bmatrix} 1 & 3 & -3 \\ -1 & -3 & 1 \\ 2 & 6 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

rows with pivots

$$\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\dim(\text{Row}(A)) = 2 \longleftarrow$ dimension of the row space

Matrix Rank

* the rank of a matrix is the number of linearly independent columns.

- $r = \text{rank}(A) = \dim(\text{Col}(A))$
- $\dim(\text{Col}(A)) = \dim(\text{Row}(A))$

Matrix Transposition

* The transpose of a matrix is just making the columns as rows in the transposed matrix.

- Example:

$$A = \begin{bmatrix} 1 & 3 & -3 \\ -1 & -3 & 1 \\ 2 & 6 & -2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ -3 & 1 & -2 \end{bmatrix}$$

* Column and Row Spaces of A^T

$$\xrightarrow{\text{REF}} \begin{bmatrix} \boxed{1} & \boxed{-1} & 2 \\ \boxed{0} & \boxed{-2} & 4 \\ \boxed{0} & \boxed{0} & 0 \end{bmatrix}$$

columns 1 & 2 of A^T are linearly independent

$$\text{Col}(A^T) = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}\right\} \xrightarrow{\text{in the}} \text{Row}(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\dim(\text{Col}(A^T)) = 2$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1 \quad 0 \quad 0} \\ \boxed{0 \quad 1 \quad -2} \\ 0 \quad 0 \quad 0 \end{bmatrix}$$

$$\text{Row}(A^T) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\} \xrightarrow{\text{in the}} \text{Col}(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}\right\}$$

$$Row(A^T) = Span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}\right\} \xrightarrow{\text{in the}} Col(A) = Span\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}-3\\1\\-2\end{bmatrix}\right\}$$

$$dim\left(Row(A^T)\right) = 2$$

Solutions of Linear Equations

* Given $A\vec{x} = \vec{b}$, the system has at least one solution if $\vec{b}$ is in the $Col(A)$.

Exampe:

$$A = \begin{bmatrix}1 & 3 & -3\\-1 & -3 & 1\\2 & 6 & -2\end{bmatrix}, \quad \vec{b} = \begin{bmatrix}2\\-2\\4\end{bmatrix}$$

$$Col(A) = Span\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}-3\\1\\-2\end{bmatrix}\right\} = \underbrace{c_1\begin{bmatrix}1\\-1\\2\end{bmatrix} + c_2\begin{bmatrix}-3\\1\\-2\end{bmatrix} = \begin{bmatrix}2\\-2\\4\end{bmatrix}}_{\text{true if } c_1 = 2 \text{ and } c_2 = 0}$$

So, $\vec{b}$ is in the $Col(A)$. $A\vec{x} = \vec{b}$ has at least one solution.

Column and Row Spaces are Vector Spaces

* Vector Space
  - A set closed under addition and scalar multiplication.
  - Column and Row Spaces are vector spaces that capture how A transforms an input vector $\vec{x}$ into $\vec{b}$.

* Column Space
  - $Col(A) = Span\{\text{columns of } A\}$
  - If A is $m \times n$, then $Col(A) \in \mathbb{R}^m$
  - A has rank $r \leq dim(Col(A))$
  - It is a subspace of the outputs; $\vec{b} \in \mathbb{R}^m$

* Row Space
  - $Row(A) = Span\{\text{rows of RREF } A\}$

* Row Space
  - $Row(A) = Span\{\text{rows of RREF } A\}$
  - If A is $m \times n$, then $Row(A) \in \mathbb{R}^n$
  - It is a subspace of the inputs; $\vec{x} \in \mathbb{R}^n$