Vectors

Objectives:

- Define a vector space
- How to test a set if it is a vector space
- Layout properties of a vector space

Motivation:

- vectors in $\mathbb{R}^n$ ($\mathbb{R}^2$ & $\mathbb{R}^3$).
- Working with functions with vectors in a structured way.
- A vector space is a general framework this structure.

Vector Space

A vector space (over $\mathbb{R}$) consist of a set V along with two operations "+" and "·" subject to conditions for all vectors in V and scalars in $\mathbb{R}$.

Example 1: Line through the origin ($\mathbb{R}^2$).

Vector space $\longrightarrow L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \boxed{y = 3x} \right\}$ ← A set of all vectors that form a line $y=3x$.

restriction

Is this a vector space?

* Define "+" and "·"

- Let $\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, and $\vec{v_3} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$ $\longrightarrow$ $y_1 = 3x_1$ ✓, $y_2 = 3x_2$, $y_3 = 3x_3$
- "+" $\longrightarrow \vec{v_1} + \vec{v_2}$
- "·" $\longrightarrow c \cdot \vec{v_1}$ or $c\vec{v_2}$

* Conditions: (1) closed under addition

$$\vec{v_1} + \vec{v_2} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \longrightarrow \begin{aligned} (y_1 + y_2) &= 3(x_1 + x_2) \checkmark \\ y_1 + y_2 &= 3x_1 + 3x_2 \end{aligned}$$

↳ adding $y_1 = 3x_1$ and $y_2 = 3x_2$

(2) Commutative

$$\vec{v_1} + \vec{v_2} = \vec{v_2} + \vec{v_1} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 + x_1 \\ y_2 + y_1 \end{bmatrix} \longrightarrow (y_2 + y_1) = 3(x_2 + x_1) \checkmark$$

the same as $(y_1 + y_2) = 3(x_1 + x_2)$

(3) Associative

$$(\vec{v_1} + \vec{v_2}) + \vec{v_3} = \vec{v_1} + (\vec{v_2} + \vec{v_3}) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + (x_2 + x_3) \\ y_1 + (y_2 + y_3) \end{bmatrix}$$

$y_1 + (y_2 + y_3) = 3(x_1 + (x_2 + x_3))$

the same as

$(y_1 + y_2) + y_3 = 3((x_1 + x_2) + x_3)$

(4) Adding $\vec{0}$ returns itself

$$\vec{v_1} + \vec{0} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \checkmark$$

(5) There exist an additive inverse so that

$$\vec{v_1} + \underbrace{\boxed{(-\vec{v_1})}}_{\text{additive inverse}} = 0 \longrightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} -x_1 \\ -y_1 \end{bmatrix} = \begin{bmatrix} x_1 + (-x_1) \\ y_1 + (-y_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow 0 = 3(0) \checkmark$$

$$\vec{v_1} + \boxed{(-\vec{v_1})} = 0 \longrightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} -x_1 \\ -y_1 \end{bmatrix} = \begin{bmatrix} x_1 + (-x_1) \\ y_1 + (-y_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow 0 = 3(0) \ \checkmark$$

additive inverse of $\vec{v_1}$

(6) Closed under scalar multiplication

$$c \cdot \vec{v_1} = c \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix} \longrightarrow cy_1 = 3cx_1$$

$$\downarrow$$

$$y_1 = 3x_1 \ \checkmark$$

(7) Adding scalars (distributing the vector)

$$(c+d)\cdot\vec{v_1} = c\vec{v_1} + d\vec{v_1} = c\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + d\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cx_1 + dx_1 \\ cy_1 + dy_1 \end{bmatrix} \longrightarrow cy_1 + dy_1 = 3(cx_1 + dx_1)$$

$$(c+d)y_1 = 3(c+d)x_1$$

$$\downarrow$$

$$y_1 = 3x_1 \ \checkmark$$

(8) Adding vectors (distributing a constant)

$$c\cdot(\vec{v_1} + \vec{v_2}) = c\cdot\vec{v_1} + c\cdot\vec{v_2} = c\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} cx_1 + cx_2 \\ cy_1 + cy_2 \end{bmatrix} \longrightarrow (cy_1 + cy_2) = 3(cx_1 + cx_2)$$

$$c(y_1 + y_2) = 3c(x_1 + x_2)$$

$$\downarrow$$

$$(y_1 + y_2) = 3(x_1 + x_2) \ \checkmark$$

(9) multiplying scalars

$$(cd)\cdot\vec{v_1} = c(d\cdot\vec{v_1}) = c\left(d\cdot\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = \begin{bmatrix} c(dx_1) \\ c(dy_1) \end{bmatrix} \longrightarrow c(dy_1) = 3c(dx_1) \ \checkmark$$

$$\downarrow$$

$$y_1 = 3x_1$$

(10) multiplying by 1.

$$1\cdot\vec{v_1} = \vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \ \checkmark$$

<u>Example 2</u>: Polynomials

Consider $P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, \dots, a_3 \in \mathbb{R}\}$

* Define "+" and "."

- In $\mathbb{R}^4$, $a_0 + a_0x + a_2x^2 + a_3x^3 \longrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$

- "+" : $(a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0x + b_1x + b_2x^2 + b_3x^3) = (a_0+b_0)x + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3$

$$\hookrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

- ".": $c\cdot(a_0 + a_1x + a_2x^2 + a_3x^3) = (ca_0) + (ca_1)x + (ca_2)x^2 + (ca_3)x^3$

$$\hookrightarrow c\cdot\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} ca_0 \\ ca_1 \end{bmatrix}$$

$$\hookrightarrow c \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} ca_0 \\ ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}$$

* Conditions ⟶ all satisfied because the coefficients are a linear combination.

Example 3: What is not a vector space?

$$V = \{ax^2 + bx + c \mid \boxed{a \neq 0}\}$$ with usual add. and scalar mult.

restriction

* Not closed under addition:
  Counter example: $(x^2+2x+1) + (-x^2+3x+2) = 5x+3$

$$\hookrightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \end{array} \begin{array}{l} a \longrightarrow a = 0 \longrightarrow \text{restriction violated} \\ b \\ c \end{array}$$

* Can't be multiplied by 0 because $0(ax^2+bx+c) = \cancel{0a}^{0}x^2 + \cancel{0b}^{0}x + \cancel{0c}^{0}$

$0 \longrightarrow$ restriction violated