Central Limit Theorem

Thursday, December 1, 2022

Normal Random Variable

pdf:

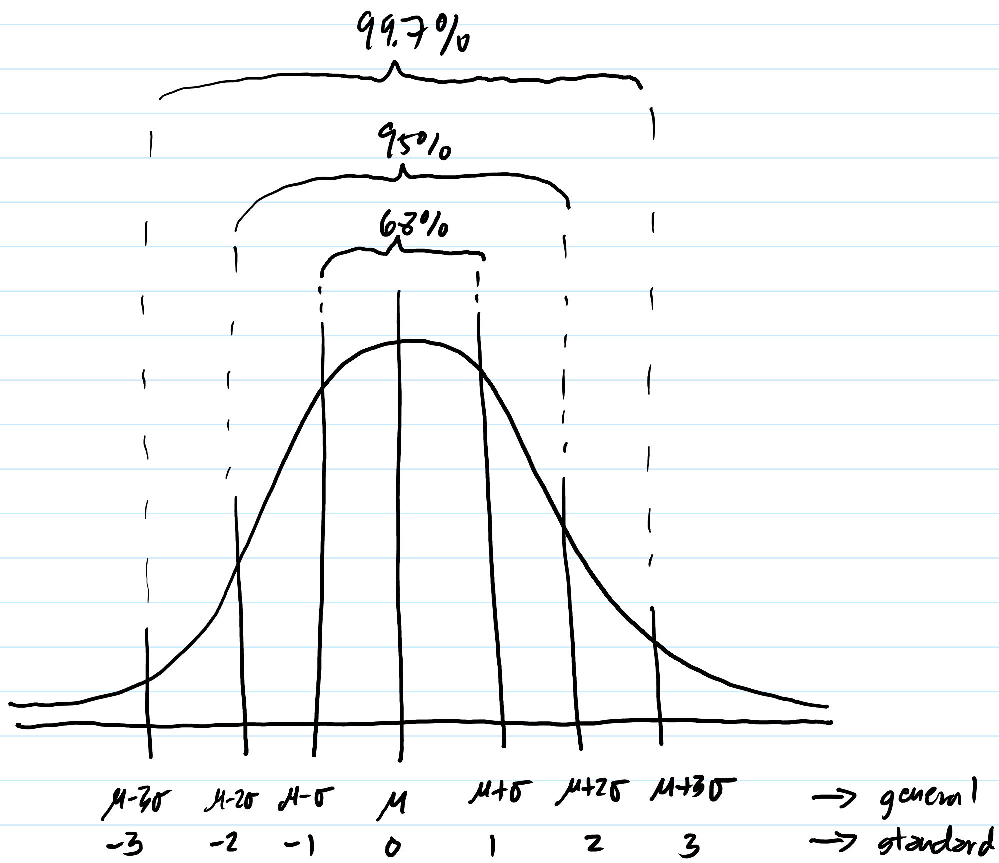
$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

cdf:

$$F(x;\mu,\sigma) = \int_{-\infty}^{x} f(t)\,dt$$

$$E[X] = \mu$$
$$Var[X] = \sigma^2$$

If $\mu = 0$ & $\sigma = 1$, then we call it the standard normal distribution.



Transformation to standard normal:

let $z = \frac{x-\mu}{\sigma}$.

pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

cdf:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Central Limit Theorem

Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be i.i.d. rv.
with $E[X_i] = \mu < \infty$ and variance $0 < Var(X_i) = \sigma^2 < \infty$.
Then, the random variable

$$Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n}\,\sigma}$$

convergerges in distribution to the standard normal random variable as n goes to infinity, that is

$$\lim_{n \to \infty} P(Z_n \le x) = \Phi(x), \quad \text{for all } x \in \mathbb{R},$$

where $\Phi(x)$ is the standard normal CDF.

Application of the CLT

1. Let X_i be random variable the represent the servic time for a customer i with mean $E[X_i] = 2$ and $Var(X_i) = 1$.

   Assume that service times for different customers are i.i.d.

   Let Y be the total time the bank teller spends serving 50 customers.

   Find $P(90 < Y < 110)$.

Solution:

$$Y = X_1 + X_2 + \cdots + X_n,$$

where
$n = 50$, $E[X_i] = \mu = 2$, and $Var(X_i) = \sigma^2 = 1$.

$$P(90 < Y < 110) = P\left(\frac{90 - n\mu}{\sqrt{n}\,\sigma} < \frac{Y - n\mu}{\sqrt{n}\,\sigma} < \frac{110 - n\mu}{\sqrt{n}\,\sigma} \right)$$

$$= P\left(\frac{90 - 100}{\sqrt{50}} < \frac{Y - n\mu}{\sqrt{n}\,\sigma} < \frac{110 - 100}{\sqrt{50}} \right)$$

$$= P\left(-\sqrt{2} < \frac{Y - n\mu}{\sqrt{n}\,\sigma} < \sqrt{2}\right)$$

By CLT, $\frac{Y - n\mu}{\sqrt{n}\,\sigma}$ is approx. standard normal

$$P(90 < Y < 100) \approx \underbrace{\Phi(\sqrt{2}) - \Phi(-\sqrt{2})}$$

using R

$$\approx \text{pnorm}(\sqrt{2}, 0, 1) - \text{pnorm}(-\sqrt{2}, 0, 1)$$

$$\approx 0.8427$$