Conditioning & Independence for CRVs

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Recall:

- Conditional Probability

definition $$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

bayes' rule $$P(B|A) = \frac{P(A|B)P(B)}{P(A)}, \quad P(A) > 0$$

- Independence

1. $P(A|B) = P(A)$

2. $\underset{\text{joint pdf}}{f_{XY}(x,y)} = \underbrace{f_X(x) f_Y(x)}_{\text{marginal pdfs}}$

3. $Cov(X,Y) = E[XY] - E[X]E[Y] = 0$

- Expected values and variance

$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx \rightarrow$ first moment

$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx \rightarrow$ 2nd moment

$M_X(s) = E[e^{sX}] = \int_{-\infty}^{\infty} e^{sX} f(x)\, dx \rightarrow$ moment generating function

$E[X^k] = \frac{d^k}{ds^k} M_X(s) \Big|_{s=0} \rightarrow$ kth moment

- Joint distributions for X & Y random variables.

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)\, dx\, dy$$

Conditional pdf and cdf for one random variable

Suppose X is a cont. rv.
Given that $X \in I = [a,b]$.

Suppose $X \in I = [a,b]$ has already occured, event A.

cumulative distribution function

cdf:

$$F_{X|A}(x) = \begin{cases} 1 & , x > b \\ \dfrac{F_X(x) - F_X(a)}{\underbrace{F_X(b) - F_X(a)}_{P(A)}} & , a \le x < b \\ 0 & , x < a \end{cases}$$

probability density function

pdf:

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{P(A)} & , a \le x < b \\ 0 & , \text{otherwise} \end{cases}$$

Conditional expectation and variance:

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\,dx$$

$$E[X^2|A] = \int_{-\infty}^{\infty} x^2 f_{X|A}(x)\,dx$$

$$Var(X|A) = E[X^2|A] - (E[X|A])^2$$

Example:

Let $X \sim Exp(1)$

↳ exponential distribution with parameter $\lambda=1$.

we know the pdf, cdf of $Exp(1)$.

$$f_X(x) = e^{-x} \quad , \quad 0 \le x < \infty$$

$$F_X(x) = \int_0^x e^{-t}\,dt$$

$$= -e^{-t}\Big|_0^x$$

$$= \frac{-1}{e^x} + \frac{1}{e^0}$$

$$= -e^{-x} + 1$$

$$F_X(x) = 1 - e^{-x}$$

a. Find the conditional pdf and cdf of X given $X>1$.

Let A to be the given, $X>1$.

$$P(A) = P(X>1) = \int_1^\infty f_X(x)\,dx$$

$$= \int_1^\infty e^{-x}\,dx$$

$$P(A) = \frac{1}{e} = e^{-1}$$

cond. pdf:

$$f_{X|A}(x) = \frac{f_X(x)}{P(A)} = \frac{e^{-x}}{e^{-1}} = e^{-x+1}$$

$$f_{X|A}(x) = \frac{f_X(x)}{P(A)} = \frac{e^{-x}}{e^{-1}} = e^{-x+1}$$

$$= \begin{cases} e^{-x+1}, & x > 1 \\ 0, & \text{otherwise} \end{cases}$$

cond cdf:

$$F_{X|A}(x) = \frac{F_X(x) - F_X(1)}{P(A)}$$

$$= \frac{(1-e^{-x}) - (1-e^{-1})}{e^{-1}}$$

$$= 1 - e^{-x+1}$$

$$F_{X|A}(x) = \begin{cases} 1-e^{-x+1}, & x > 1 \\ 0, & \text{otherwise} \end{cases}$$

b. Find $E[X|A]$.

$$E[X|A] = \int_1^\infty x f_{X|A}(x)\,dx$$

$$= \int_1^\infty x e^{-x+1}\,dx$$

$$= e \int_1^\infty x e^{-x}\,dx$$ → use IBP

$$= e\left[-e^{-x} - xe^{-x}\right]\Big|^\infty$$

$$= e\left[-e^{-x} - xe^{-x}\right]\Big|_1^{\infty}$$

$$= e\frac{2}{e}$$

$$E[X|A] = 2$$

b. Find $Var(X|A)$.

$$Var(X|A) = E[X^2|A] - (E[X|A])^2$$

$$E[X^2|A] = \int_1^{\infty} x^2 f_{X|A}(x)\,dx$$

$$= \int_1^{\infty} x^2 e^{-x+1}\,dx$$

$$= e\int_1^{\infty} x^2 e^{-x}\,dx$$

IBP twice

$$E[X^2|A] = 5$$

$$Var(X|A) = 5 - (2)^2$$

$$= 1$$

Conditional pdf and cdf by another random variable

Suppose X & Y are jointly cont. rv.

1. conditional pdf of X given $Y=y$:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

2. finding the probability $X \in A$ given $Y=y$:

A $\downarrow$ domain region

$$P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx$$

3. conditional cdf of X given $Y=y$:

$$F_{X|Y}(x,y) = P(X \le x \mid Y=y) = \int_{-\infty}^{x} f_{X|Y}(x|y)\,dx$$

4. expected values & variance

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$

$$E[X^2|Y=y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\,dx$$

$$Var(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2$$

Example:

Suppose X & Y are jointly cont. r.v.

$$f_{XY}(x,y) = \begin{cases} \frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6}, & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Given $0 \leq y \leq 2$.

a. Find the cond. pdf of X given $Y=y$.

marginal pdf of Y.

$$f_Y(y) = \int_0^1 \left(\frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6}\right) dx$$

$$f_Y(y) = \frac{3y^2 + y + 1}{12} \, , \text{ for } 0 \leq y \leq 2$$

cond. pdf:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \frac{x^2/4 + y^2/4 + xy/6}{(3y^2+y+1)/12}$$

$$f_{X|Y}(x|y) = \frac{3x^2 + 3y^2 + 2xy}{3y^2 + y + 1} \, , \quad 0 \leq x \leq 1$$

b. Find $P(X < \frac{1}{2} \mid Y=y)$.

$$P(X < \tfrac{1}{2} \mid Y = y) = \int_0^{1/2} f_{X|Y}(x|y)\, dx$$

constant

constant

$$= \int_0^{1/2} \frac{3x^2 + 3y^2 + 2xy}{3y^2 + y + 1} dx$$

constant

constant

$$\quad 3y^2 + y + 1$$

constant constant

$$= \frac{1}{3y^2+y+1} \int_0^{1/2} (3x^2 + 3y^2 + 2xy)\, dx$$

$$P(X < \tfrac{1}{2} \mid Y = y) = \frac{(3/2)y^2 + y/4 + 1/8}{3y^2 + y + 1}$$

c. Find $E[X \mid Y=1]$ and $E[X^2 \mid Y=1]$, $Var(X \mid Y=1)$

$$E[X \mid Y=1] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|1)\, dx$$

$$= \int_0^1 x \left. \frac{3x^2 + 3y^2 + 2xy}{3y^2+y+1} \right|_{y=1} dx$$

$$= \int_0^1 x \, \frac{3x^2 + 3 + 2x}{3+1+1}\, dx$$

$$= \frac{1}{5} \int_0^1 x(3x^2 + 2x + 3)\, dx$$

$$= \frac{1}{5} \int_0^1 (3x^3 + 2x^2 + 3x)\, dx$$

$$E[X \mid Y=1] = \frac{7}{12}$$

$$E[X^2 \mid Y=1] = \int_0^1 x^2 f_{X|Y}(x|1)\, dx$$

$$= \frac{1}{5}\int_0^1 3x^4 + 2x^3 + 3x^2 \, dx$$

$$= \frac{21}{50}.$$

$$Var(X \mid Y=1) = E[X^2 \mid Y=1] - (E[X \mid Y=1])^2$$

$$= \frac{21}{50} - \left(\frac{7}{12}\right)^2$$

$$= \frac{287}{3600}$$