Covariance & Correlation for CRVs

Tuesday, November 1, 2022

Continuous Random Variables

- probability density functions. $f(x)$

  $X \to$ cont. random variable with pdf $f(x)$.

properties

1. $f(x) > 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

- To find the probability $P(X=x)$, then

  $P(X=x) = f(x)$.

- To find the probability $P(X \le x)$, then

  $$P(X \le x) = \int_{-\infty}^{x} f(x)\,dx.$$

- To find the probability of $P(a \le X \le b)$, then

  $$P(a \le X \le b) = \int_{a}^{b} f(x)\,dx.$$

- Kth moment, expected values, and variance

  1st moment (mean): $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$

  2nd moment: $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

  moment generating functions: $M_X(s) = E[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} f(x)\,dx$

  Kth moment: $E[X^k] = \frac{d^k}{ds^k} M_X(s)\Big|_{s=0}$

Variance: $Var(X) = E[X^2] - (E[X])^2$

std: $\sigma(X) = \sqrt{Var(X)}$

Joint and Marginal distributions

Suppose X and Y are jointly continuous random variables.

$$P((X,Y)\in A) = \iint_A \underbrace{f_{XY}(x,y)}_{\text{joint pdf}}\,dx\,dy$$

with domain $f_{XY}: \mathbb{R}^2 \to \mathbb{R}$ so that $A \in \mathbb{R}^2$.

Properties:

1. $\{(x,y) \mid f_{XY}(x,y) > 0\}$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

Marginal densities

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

Independence: $f_{XY}(x,y) = f_X(x) f_Y(y)$.

Covariance & Correlation

for two random variable X & Y,

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

covariance $Cov(X,Y) = E[XY] - E[X]E[Y]$

If X & Y are independent, then $Cov(X,Y) = 0$

correlation

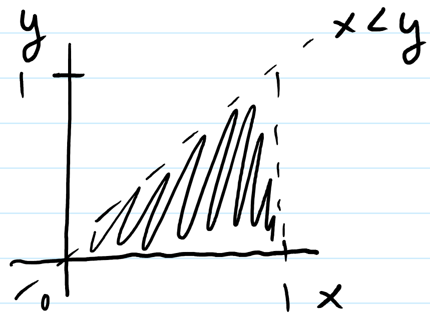
$$\rho(x,y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}, \quad -1 \le \rho \le 1$$

- if $\rho = 0$, no correlation
- if $\rho > 0$, positive correlation
- if $\rho < 0$, negative correlation

Example:

Suppose X and Y are continuous random variables with joint pdf

$$f_{XY}(x,y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$



Way 1:

marginal distributions

$$f_X(x) = \int_x^1 8xy\,dy$$

$$= 8x\frac{y^2}{2}\Big|_x^1$$

$$= 4x - 4x^3$$

$$f_X(x) = 4x(1-x^2), \quad 0 < x < 1$$

$$f_Y(y) = \int_0^y 8xy\,dx$$

$$= \frac{8x^2}{2}y\Big|_0^y$$

$$f_Y(y) = 4y^3, \quad 0 < y < 1$$

$$f_{XY}(x,y) \stackrel{?}{=} f_X(x)\,f_Y(y)$$

$$8xy = (4x(x-1))(4y^3)$$

$$= 4y^3(4x^2 - 4x)$$

$$8xy \neq 16y^3x^2 - 16y^3x$$

Therefore X & Y are not independent.

Way 2: Check if $Cov(X,Y) \neq 0$.

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

- $E[X] = \int_0^1 x(4x(1-x^2))\,dx$

$$= \int_0^1 (4x^2 - 4x^4)\,dx$$

$$= \frac{4x^3}{3} - \frac{4x^5}{5}\Big|_0^1$$

$$E[X] = \frac{4}{3} - \frac{4}{5} = \frac{8}{15}$$

- $E[Y] = \int_0^1 y(4y^3)\,dy$

$$= \int_0^1 4y^4\,dy$$

$$= \frac{4y^5}{5}\Big|_0^1$$

$$E[Y] = \frac{4}{5}$$

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- $E[XY] = \int_0^1 \int_0^1 8xy \, dx \, dy$

$$= \int_0^1 \left(4x^2 y \Big|_0^1 \right) dy$$

$$= \int_0^1 4y \, dy$$

$$= \frac{4y^2}{2} \Big|_0^1$$

$E[XY] = 2$

- $E[XY] = E[X]E[Y]$

$$2 = \left(\frac{8}{15}\right)\left(\frac{4}{5}\right)$$

$$2 \neq \frac{32}{75}$$

Thus, X & Y are not independent.