Law of Large Numbers

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Definition:

Given independent and Identically Distributed iid random variables $X_1, X_2, \dots, X_n$, the <u>sample mean</u> denoted $\bar{X}$ is defined as

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}.$$

We can also write M_n.

Since X_i's are random variables, the sample mean $\bar{X} = M_n(X)$ is also a random variable.

Expected value:

$$E[\bar{X}] = \frac{E[X_1] + E[X_2] + \dots + E[X_n]}{n} \rightarrow \text{(by linearity of Expectation)}$$

$$= \frac{nE[X]}{n} \rightarrow \text{Since } E[X_i] = E[X]$$

$$E[\bar{X}] = E[X]$$

Variance:

$$Var(\bar{X}) = \frac{Var(X_1 + X_2 + \dots + X_n)}{n^2} \rightarrow \text{since } Var(aX) = a^2 Var(X)$$

$$= \frac{Var(X_1) + Var(X_2) + \dots + Var(X_n)}{n^2} \rightarrow \text{since } X_i\text{'s are ind.}$$

$$= \frac{n\,Var(X)}{n^2} \rightarrow \text{Since } Var(X_i) = Var(X)$$

$$Var(\bar{X}) = \frac{Var(X)}{n}$$

<u>Weak law of large numbers.</u>

Let $X_1, X_2, \dots, X_n$ be iid rv with finite expected value $E[X_i] = \mu < \infty$. then, for any $\epsilon > 0$

$$\lim_{n \to \infty} P(|\bar{X} - \mu| \geq \epsilon) = 0$$

Proof: Assume that $Var(X) = \sigma^2$ is finite.
Using Chebyshev's inequality

$$P(|\overline{X} - \mu| \geq \epsilon) \leq \frac{Var(\overline{X})}{\epsilon^2}$$

$$= \frac{Var(X)}{n\epsilon^2}.$$

So, $\lim_{n\to\infty} \frac{Var(X)}{n\epsilon^2} = 0.$

Thus, $\lim_{n\to\infty} P(|\overline{X} - \mu| \geq \epsilon) = 0.$

<u>Strong law of large numbers</u>

Suppose that the first moment $E[X]$ of X is finite.
then,
$\overline{X}_n$ converges in probability to $E[X]$,
thus $\lim_{n\to\infty} P(|\overline{X}_n - E[X]| \geq \epsilon) = 0$ for every $\epsilon > 0$.

<u>Chebyshev's Inequality</u>

Let X be a random variable and $a \in \mathbb{R}^+$.
Assume X has density function f_X, then

$$E[X^2] = \int_{\mathbb{R}} x^2 f_X(x)\,dx$$

$$\geq \int_{|x|\geq a} x^2 f_X(x)\,dx$$

$$\geq a^2 \int_{|x|\geq a} f_X(x)\,dx = a^2 P(|X| \geq a)$$

So, $P(|X| \geq a) \leq \frac{1}{a^2} E[X^2]$

In general, $P(|X| \geq a) \leq \frac{1}{a^p} E(|X|^p)$ for any $p = 1, 2, \ldots$

$E(|x|^p)$ is finite