Law of Total Probability for CRVs

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Law of total probability

- Discrete:

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i)P(B_i)$$

- Continuous:

$$P(A) = \int_{-\infty}^{\infty} P(A|X=x) f_X(x)\, dx$$

Law of total expectation

- continuous:

$$E[Y] = \int_{-\infty}^{\infty} E[Y|X=x] f_X(x)\, dx$$

$$= E\left[E[Y|X]\right]$$

Law of total variance

- Continuous:

$$Var(Y) = E[Var(Y|X)] + Var(E[Y|X])$$

Example:

Let X and Y be two independent Uniform$(0,1)$ random variables. Find $P(X^3+Y>1)$, $E[Y]$, and $Var(Y)$.

$$P(X^3+Y>1) = \int_{-\infty}^{\infty} P(X^3+Y>1 \mid X=x) f_X(x)\, dx$$

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uniform distribution $\frac{1}{b-a} = \frac{1}{2-1} = \frac{1}{1} = 1$

$$= \int_0^1 P(X^3+Y>1 \mid X=x)(1)\,dx$$

$$= \int_0^1 P(Y>1-x^3)\,dx$$

X & Y are independent

$$= \int_0^1 x^3\,dx$$

Y is Uniform$(0,1)$

$$P(X^3+Y>1) = \frac{1}{4}$$

$$E[Y] = \int_{-\infty}^{\infty} E[Y \mid X=x] f_X(x)\,dx$$

expected value of the uniform dist is $\frac{b-a}{1} = \frac{2-1}{1} = 1$

$$= \int_0^1 (1)(1)\,dx$$

$$= \int_0^1 dx$$

$$E[Y] = 1$$

$$Var(Y) = E[Var(Y \mid X)] + Var(E[Y \mid X])$$

we know $E[Y \mid X] = 1 = E[Y]$ → because X & Y are independent

$$Var(Y \mid X) = \frac{(b-a)^2}{12} = \frac{1}{12} = Var(Y)$$

$$Var(Y) = E\left[\frac{1}{12}\right] + Var(1)$$

$$= \frac{1}{} + 0$$

$$= \frac{1}{12} + 0$$

$$Var(Y) = \frac{1}{12}$$