Moment Generating Functions for CRVs

Tuesday, November 15, 2022

The moment generating function (mgf) of a random variable X is a function $M_X(s)$ defined as

$$M_X(s) = E\left[e^{sX}\right].$$

The mgf exists if there exists a positive constant a such that $M_X(s)$ is finite for all $s \in [-a, a]$.

- Discrete case:

(mgf) $$M_X(s) = \sum_x E[X^k] \frac{s^k}{k!}$$

(moments) $$E[X^k] = \frac{d^k}{ds^k} M_X(s) \Big|_{s=0}$$

or

$$E[X^k] = \sum_x x^k p(x)$$

- Continuous case:

(mgf) $$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f(x)\, dx$$

(moments) $$E[X^k] = \frac{d^k}{ds^k} M_X(s) \Big|_{s=0}$$

or

$$E[X^k] = \int_{-\infty}^{\infty} x^k f(x)\, dx$$

Examples of known mgfs of common pdfs & pmfs.

Bernoulli: $M_X(s) = e^s p + 1 - p$ → discrete

Binomial: $M_X(s) = (pe^s + (1-p))^n$ → discrete

Poisson: $M_X(s) = e^{\lambda(e^s - 1)}$ → discrete

Exponential: $M_X(s) = \dfrac{\lambda}{\lambda - s}$ → cont.

Standard normal: $M_Z(s) = e^{s^2/2}$ → cont.

$\downarrow$

$\mu = 0, \ \sigma = 1$

General normal: $M_X(s) = e^{s\mu + t^2\sigma^2/2}$ → cont.

Proposition 1

Suppose $X_1, \dots, X_n$ are n independent random variables and

$$Y = X_1 + \dots + X_n.$$

then

$$M_Y(s) = M_{X_1}(s) \cdot \ldots \cdot M_{X_n}(s).$$

Proof:

Since all $X_1, \dots, X_n$ are independent, then

$$\begin{aligned} M_Y(s) &= E\left[e^{s(X_1 + \dots + X_n)}\right] \\ &= E\left[e^{sX_1 + \dots + sX_n}\right] \\ &= E\left[e^{sX_1} \cdot \ldots \cdot e^{sX_n}\right] \\ &= E\left[e^{sX_1}\right] \cdot \ldots \cdot E\left[e^{sX_n}\right] \end{aligned}$$

$$M_Y(s) = M_{X_1}(s) \cdot \dots \cdot M_{X_n}(s)$$

Proposition 2

Suppose for two random variables X & Y we have $M_X(s) = M_Y(t) < \infty$ for all t in an interval, then X and Y have the same distribution.

Joint mgf

the Joint mgf of X and Y is

$$M_{X,Y}(s,t) = E\left[e^{sX+tY}\right].$$

If X & Y are independent, then

$$M_{X,Y}(s,t) = M_X(s) M_Y(t).$$

Recall Convolution formula

We need the pmf or pdf of $Z = X+Y$.

- Discrete case : $P_Z(z) = \sum_{x \in X} P_X(x) P_Y(z-x)$
- continuous case : $f_Z(z) = \int_{x \in X} f_X(x) f_Y(z-x) dx$

Example:

1. Suppose that mgf of X is given by $M_X(s) = e^{3(e^s-1)}$. Find $P(X=0)$.

this looks like a poisson mgf.

Observe: $M_X(s) = e^{3(e^s-1)} = e^{\lambda(e^s-1)}$, where $\lambda = 3$.

So, $X \sim \text{Poisson}(3)$.

pmf of poisson: $P(X=x;\lambda) = e^{-\lambda}\frac{\lambda^x}{x!}$

So, since $\lambda = 3$,

$$P(X=0;3) = e^{-3}\frac{3^0}{0!} = e^{-3}.$$

2. Suppose X has the mgf

$$M_X(s) = (1-2s)^{-1/2} \text{ for } s < \frac{1}{2}.$$

Find the 1st and 2nd moments of X.

$$M_X'(s) = -\frac{1}{2}(1-2s)^{-3/2}(-2) = (1-2s)^{-3/2}$$

$$M_X''(s) = -\frac{3}{2}(1-2s)^{-5/2}(-2) = 3(1-2s)^{-5/2}$$

$$E[X] = M_X'(s)\Big|_{s=0} = 1.$$

$$E[X^2] = M_X''(s)\Big|_{s=0} = 3.$$